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Modeling Radical Indeterminacy in Branching Time Semantics

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Abstract

With the primary aim of preserving indeterminism and keeping the future open, several different branching time semantics have been proposed. The temporal frames on which these semantics are based are trees whose branches are typically endless. However, it is a genuine metaphysical possibility that the series of moments might come to an end. In light of this, a radical notion of indeterminacy can be adopted, one that is formally captured not merely by the branching of the future, but by a tree-like structure of time that, in addition to unbounded branches, also includes branches that terminate (dead branches). In this article, (i) we show how such a temporal frame can be formally developed, (ii) examine some implications of incorporating dead branches for both Peircean and Ockhamist branching time semantics, and (iii) outline a new semantics tailored to this kind of trees.

Keywords: Radical Indeterminacy, Branching Time Semantics, Doomsday, End of Time

1 Introduction

Branching time semantics (BTSs) are formal semantics for temporal logics[1] specifically designed to model an indeterministic view of temporal reality—that is, to capture the idea of an open future. They build on the Kripkean framework of modal logics, replacing the set of possible worlds with a non-empty set T

[1] Among others, see e.g. Goranko (2023) for a clear and comprehensive introduction to temporal logics, including BTSs.

of moments (or time instants)[2] and the accessibility relation with a temporal precedence relation $<$.[3] The structure $\mathcal{T} = \langle T, < \rangle$ is called a *temporal frame*. When it is supplemented with a suitable valuation function V—which assigns an appropriate truth value to the propositional variables of the chosen language $\mathcal{L}$—it yields a so-called *temporal model*, $\mathcal{M} = \langle T, <, V \rangle$. Then, the truth of an arbitrary formula ϕ of $\mathcal{L}$ at a given moment t in the model $\mathcal{M}$ (i.e., $\mathcal{M}, t \models \phi$) is recursively defined via semantic clauses for the usual logical connectives and for the tense operators of $\mathcal{L}$. Finally, the notions of validity in a model ($\mathcal{M} \models \phi$), validity in a frame ($\mathcal{T} \models \phi$), validity ($\models \phi$), and logical consequence ($\Sigma \models \phi$) are given.[4]

The 'recipe' briefly sketched above does not, by itself, yield a BTS. Crucially, different formal constraints can be imposed on $\mathcal{T}$, and depending on which ones are chosen, one obtains different types of temporal frames—and, with them, different semantics. BTSs are essentially those semantics that rely on frames known as *trees*, that is, structures where $<$ satisfies the following conditions:

(Transitivity) $\quad \forall t_1 \forall t_2 \forall t_3 ((t_1 < t_2 \wedge t_2 < t_3) \rightarrow t_1 < t_3)$

(Irreflexivity) $\quad \forall t \neg (t < t)$

(Backward-linearity, BL) $\quad \forall t_1 \forall t_2 \forall t_3 ((t_1 < t_3 \wedge t_2 < t_3) \rightarrow$
$\rightarrow (t_1 < t_2 \vee t_2 < t_1 \vee t_1 = t_2))$

(Historical Connectedness, HC) $\quad \forall t_2 \forall t_3 \exists t_1 (t_1 \leqslant t_2 \wedge t_1 \leqslant t_3)$[5]

Transitivity and Irreflexivity make $<$ a *strict partial order*, and guarantee Asymmetry: $\forall t_1 \forall t_2 \neg (t_1 < t_2 \wedge t_2 < t_1)$. BL ensures that any two instants have a common $<$-predecessor in T, thereby ruling out backward branching. HC requires that any two instants lie on histories that intersect in the past. By imposing these conditions, we obtain a tree structure $\mathcal{T}$, as exemplified in Figure 1. One notable aspect of $\mathcal{T}$ is its branching into multiple histories.[6] A *history* h is a *maximal* set of time instants $t \in T$ that is linearly ordered by $<$ (i.e., a maximal linear subset of T). More precisely, for any $h \subseteq T$, h is a history iff $<$ satisfies the two following conditions:

(Linearity) $\quad \forall t_1 \forall t_2 ((t_1 \in h \wedge t_2 \in h) \rightarrow (t_1 < t_2 \vee t_2 < t_1 \vee t_1 = t_2))$

[2]In the context of BTSs, and in particular of so-called *synchronized* trees, it is possible to distinguish between moments and instants. See, for instance, Spolaore and Zanardo (2025, §2.1). For the purposes of this work, however, this distinction is not relevant, and the two terms will be used interchangeably.

[3]Here, for the sake of simplicity, we focus on the more common instant-based semantics, setting aside the alternative approach of interval-based semantics.

[4]What we have just outlined is the standard approach to BTSs. It should be noted, however, that alternative and highly relevant approaches are available, such as the *topological* (see e.g. Sabbadin and Zanardo (2003)) and the *geometrical* ones (see e.g. van Benthem (1999)). The latter will be discussed in §3.

[5]As usual, $t_1 \leqslant t_2 =_{def} t_1 < t_2 \vee t_1 = t_2$. The use of $\leqslant$ rather than $<$ in defining connectedness is necessary to allow for the possibility of an initial moment—i.e., a lower bound in the series of instants.

[6]Note that a linear structure also meets the conditions outlined above, along with Forward-linearity: $\forall t_1 \forall t_2 \forall t_3 ((t_3 < t_1 \wedge t_3 < t_2) \rightarrow (t_1 < t_2 \vee t_2 < t_1 \vee t_1 = t_2))$. Therefore, strictly speaking, it constitutes a tree, although with a single history. But in this case, the ordering is no longer partial—it becomes *total*. Branching arises precisely when we insist on partiality—when some time instants remain $<$-incomparable.

(Maximality) $\forall t_1(t_1 \in h \rightarrow \forall t_2((t_1 < t_2 \vee t_2 < t_1) \rightarrow t_2 \in h))$

Linearity ensures that there is no branching within individual histories, while Maximality guarantees that histories are not $<$-extendable in T—i.e., that every history is not properly contained in any other subset of T that is also linearly ordered with respect to $<$. We denote by $\mathcal{H}(\mathcal{T})$—or simply by $\mathcal{H}$—the set of all histories of $\mathcal{T}$, and by $\mathcal{H}(t)$ the set of all histories of $\mathcal{T}$ passing through a given moment t.[7]

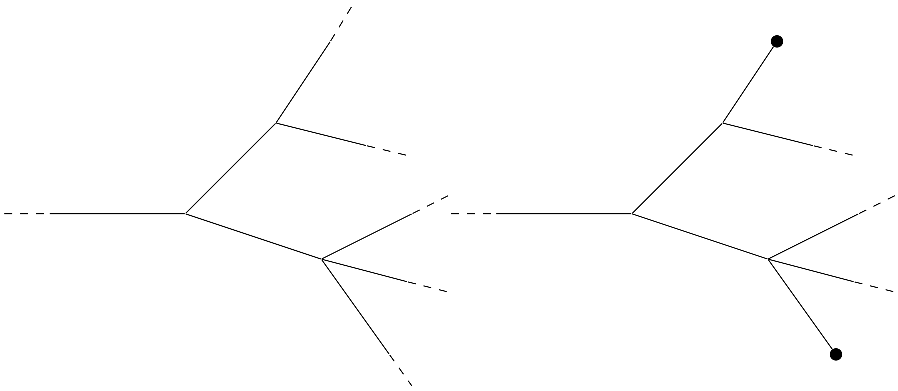

Figure 1: (Left) A tree structure $\mathcal{T}$. This particular tree also satisfies Backward-unboundedness and Forward-unboundedness (BU and FU, see below), and represents the kind of structure most commonly adopted in BTS. (Right) A different tree structure, one that satisfies BU and Forward-boundedness (FB, see below). This results in some branches of the tree being 'dead', terminating at a final moment—a doomsday—which is represented in the diagram by a terminal circular node. Note, however, that given the presence of unbounded histories, it remains possible for the future to extend indefinitely.

There are, however, additional formal constraints that can be imposed on $\mathcal{T}$ which, while preserving its tree-like structure, serve to characterize its distinctive features in greater detail. After all, there is a wide variety of trees—often quite different from one another—corresponding to different ways of modeling time. Essentially, tree structures differ along two dimensions: (i) the order-theoretic properties of time instants and (ii) the boundedness of their sequence. As for (i), one might opt for a discrete, dense, or continuous distribution of instants. While this will not concern us here, it is worth noting that the choice

[7] A brief note on terminology is in order. While some authors (e.g., Burgess (1978, p. 160)) take a branch of the tree and a history to be the same thing, others distinguish between the two. This is the case, for instance, with Goranko (2023, p. 47), for whom a history is defined as above, while a branch is identified with the ordered pair (h, t)—that is, the *part* (i.e., the subset) of the history consisting of the moments that are equal to or later than t. Accordingly, under this perspective a history is just a maximal branch. In this work, we will adopt this distinction.

influences the cardinality of T: density calls at least for countably many instants, continuity for uncountably many. Moreover, the size of T also affects (ii). Recall the following conditions:

(Backward-boundedness, BB) $\exists t_1 \neg \exists t_2 (t_2 < t_1)$

(Forward-boundedness, FB) $\exists t_1 \neg \exists t_2 (t_1 < t_2)$

BB means that $\mathcal{T}$ admits a beginning of time—i.e., its trunk ends in the past—and accordingly, so does every history. On the other hand, FB means that some history ends—i.e., it has a final moment—though whether the others do the same or go on indefinitely is left open (see Figure 1, right). Alternatively, one could consider trees that meet the opposite conditions:

(Backward-unboundedness, BU) $\forall t_1 \exists t_2 (t_2 < t_1)$

(Forward-unboundedness, FU) $\forall t_1 \exists t_2 (t_1 < t_2)$

For $\mathcal{T}$ to be backward-unbounded means that there is no beginning—i.e., the trunk stretches indefinitely into the past. Being forward-unbounded means that there is no end, which implies—and this will be crucial for what follows—that none of the branches ever terminates (as in Figure 1, left).

Now, a few remarks. First, FU has the effect of regimenting all histories in the same way, by making them all unbounded. FB, by contrast, forces at least one history to be bounded, while leaving the status of the others—i.e., whether bounded or unbounded—indeterminate. Thus, mixed cases where some branches terminate while others extend indefinitely are ruled out by FU, whereas they are allowed under FB, though not guaranteed. These cases are also possible when neither FB nor FU is assumed. In this context, all options are on the table: trees whose branches are all unbounded, trees whose branches are all bounded, and trees that mix both bounded and unbounded branches. For none of these structures violates the initial conditions—namely, that $<$ is a strict partial order on T satisfying FL and HC. Then, the point we want to emphasize is that none of these three options—imposing FB, imposing FU, or imposing neither—forces trees to have both bounded and unbounded branches, which are precisely the temporal frames we are interested in. Moreover, not even the more sophisticated *bundled trees* are of help in this regard.[8] A bundled tree is a pair $(\mathcal{T}, \mathcal{B})$ where $\mathcal{T}$ is a tree and $\mathcal{B}$ is a bundle of $\mathcal{T}$, the latter being defined as a subset of $\mathcal{H}$ that satisfies the following closure property: every $t \in T$ belongs to some history from $\mathcal{B}$. These refined frames help rule out some unwanted or irrelevant histories when defining the semantic clauses for connectives and operators, leading to a bundled-tree semantics. But, crucially, the closure property for bundles prevents them from filtering histories based on boundedness, so that bundled-tree semantics face the same limitation as standard BTSs. And finally, as previously noted, the cardinality of T affects branch boundedness. If T is finite, then all of its branches must also be finite, and therefore bounded—both forward and backward.[9] By contrast, if T is infinite, at least one history h_* necessarily consists of infinitely many instants. But of course, this tells us nothing about whether h_* is bounded or unbounded—let

[8] See e.g. Burgess (1978, pp. 167-8) and Goranko (2023, p. 36) on bundled trees.

[9] This follows from the Irreflexivity of $<$, which rules out the possibility of $<$-loops.

alone what happens with the other branches. Nevertheless, if the history at stake is infinite and bounded, it must be dense; if it is infinite and discrete, it is unbounded.

Most of the literature on BTSs tends to assume—often implicitly—both BU and FU, thereby taking T to be an infinite set. This is often regarded as a natural and convenient simplification[10] and the semantic clauses for the tense operators are consequently defined over this class of tree structures. In this paper, however, we would like to explore a different scenario. The idea is to focus on trees containing both bounded (or dead) and unbounded branches, that satisfy a number of additional constraints to be specified in due course. The reason for this lies in a different kind of indeterminacy—already anticipated by Barnes and Cameron (2011)—which we may call *strong* or *radical* indeterminacy, and which we aim to model. In the context of BTSs, indeterminacy is typically understood as the branching, at a given moment, into multiple possible courses of events, where it is undetermined which one will actually be realized. What is indeterminate is which branch of the tree's crown will be taken, since "some aspects of the future are not yet inevitable consequences of things that have already occurred" (Burgess, 1978, p. 159). By contrast, under radical indeterminacy, what is at stake is not just which future will unfold, but *whether* any future will unfold at all. To quote Barnes and Cameron (2011, p. 14, emphasis in original): "[i]t seems possible that the future is open not simply in terms of *what* will happen, but whether *anything* will happen." To this end, the paper is structured into four sections. In §2, we offer some justification for adopting a branching model of time that includes both finite and infinite histories—that is, for radical indeterminacy. In §3, we show how to formally construct such structures. In §4, we discuss the implications that dead branches have for some of the most well-known BTSs—namely, the Peircean and the Ockhamist ones. Finally, in §5, we propose an alternative semantics designed to address the main drawbacks affecting the existing ones.

2 End of Time

Disturbing as it may be, the idea that time could come to an end is a possibility we cannot rule out. Since there are different kinds of possibility, this claim calls for both clarification and justification. Essentially, we argue that the end of time—a doomsday scenario—is possible in three distinct respects: it is an *epistemic* possibility, a *physical* (or nomic) possibility, and a *metaphysical* possibility. In what follows, we aim to substantiate this claim.

To begin with, cosmology still considers such a scenario a viable option for our actual world, though it is not the one most strongly supported by recent observations. Roughly, the standard cosmological model—i.e., the well-known Big Bang model—describes the evolution of the universe based on the field equations of General Relativity. Under appropriate and widely accepted assumptions (e.g., homogeneity and isotropy), these equations take the form of the Friedmann-Lemaître dynamical equation (Ryden, 2016, §4.2), which essentially depends on two free parameters: the Hubble constant, H_0, and the cosmic density parameter, Ω. While the first measures the expansion rate of the

[10] See e.g. Goranko (2023, pp. 41, 55) and Burgess (1978, p. 160, condition (iii)) as examples of this assumption.

universe, the second measures its total density of matter and energy relative to a precisely defined critical density. Since they are not subject to significant theoretical constraints (i.e., they are free in the model), these quantities must be determined empirically, indirectly through the measurement of other observable quantities, which is an extremely challenging task. Crucially, the global evolution of the universe, including its eventual fate, is tightly constrained by the value of Ω. For depending on whether Ω is less than, equal to, or greater than 1, the geometry (i.e., the curvature) of the universe changes, and with it, its dynamics. A universe with $\Omega < 1$ is said to be open, experiencing an accelerated cosmic expansion that proceeds indefinitely, thereby ensuring the endless unfolding of time.[11] If $\Omega = 1$, the universe is flat and also continues to expand, though at a constant rate. Finally, and conversely, if $\Omega > 1$ the universe is closed, so that its current expansion will eventually reverse, leading to a contraction phase. This would plausibly lead it to a singularity—a state similar to that of the initial moment—marking its end, and the end of time itself; or alternatively, to the conclusion of a cosmic cycle of expansion and contraction, after which a new cycle would begin.

Now, several comments are in order. First, as sophisticated and impressive as the standard cosmological model may be, it is just that: a model. For it is incomplete in several respects (Smeenk and Ellis, 2017, §1.4). To mention one aspect that is particularly relevant to our discussion, the standard model lacks descriptive power in the earliest stages of the universe and loses predictive power in its potential final phases. This is because, under the extreme physical conditions predicted in those regimes, the field equations of General Relativity are no longer adequate. Therefore, as long as an adequate unified theory of gravitation that remains valid under such conditions is not available, any extrapolation about what happens in those phases is essentially speculative. Second, there is broad—albeit cautious—consensus, supported by increasingly strong empirical evidence, that $\Omega \approx 1$. This means that our universe is (nearly) flat, which would rule out cosmic collapse scenarios ending in the bleak prospect of the cessation of everything—including time itself. However, great caution must be exercised. Despite the remarkable achievements of modern cosmology, the degree of accuracy required to draw definitive conclusions about the fate of our universe is still beyond current reach. Moreover, the actual components of Ω—that is, the matter-energy content of our universe, such as baryonic matter, radiation, dark matter and dark energy—have long constituted one of the deepest open problems in all of physics. So much so, in fact, that many expect major surprises ahead, potentially leading to a drastic paradigm shift, with significant implications for the standard cosmological model itself. Thus, although our best physics and the latest evidence do not, by themselves, support the hypothesis of an end of time, they also fail to rule it out: the possibility of time coming to an end in our actual world remains a live option within current physical inquiry. Accordingly, a doomsday scenario for our actual world is epistemically possible, that is, it remains consistent with all that we currently know. In addition, what we have said so far also shows that the end of time is physically possible, i.e., consistent with our laws of nature.

[11] It is within this scenario that the possibility of a heat death of the universe is considered. In such a context, it may no longer be appropriate to speak of an endless unfolding of time, since it is unclear whether time, in such a 'frozen' situation, can still be said to retain its ordinary properties. However, we shall not address this complication in the present work.

That there are physically possible worlds—i.e., worlds governed by the same physical laws as the actual one—in which time does come to an end follows directly from the standard cosmological model. Indeed, taking the field equations of General Relativity as the relevant laws, one can obtain worlds with different dynamical evolutions simply by changing the value of Ω. Among these are closed universes, in which time does have an end.

What is more, cosmology is but one part of the story. For our inquiry is grounded in metaphysical considerations, which demand a perspective that extends beyond strictly physical aspects. Now, insofar as it is physically possible, the end of time also qualifies as a metaphysical possibility. The relationship between physical and metaphysical possibility is a complex and controversial issue.[12] But at the very least, it is acknowledged that the scope of physical possibility falls within that of metaphysical possibility: if something is physically possible, then it is certainly also metaphysically possible. However, suppose, for example, that cosmologists were to gather conclusive evidence in support of a cosmological model predicting a time that never comes to an end. In other words, let us assume that the end of time proves to be physically impossible. Still, it might remain a metaphysically possible scenario. Whether metaphysical possibility entails physical possibility is a point of contention. If it does—as maintained by the so-called *necessitarians*—the two notions effectively collapse into one, at least from an extensional point of view. If it does not—as argued by *contingentists*—then metaphysical possibility extends beyond physical possibility, and certain events that are physically impossible may nonetheless remain metaphysically possible. We need not enter into the details of that debate here. What matters for present purposes is that it is the latter view—known as Contingentism—to which we appeal in defending the end of time as a metaphysical possibility, regardless of whether it is physically possible or not. After all, philosophers have considered a range of extra-mundane entities—such as evil demons—endowed with extravagant powers, including, perhaps, the ability to bring about the end of the world with a single snap of their fingers. Although such odd entities are likely physically impossible, they are nonetheless often considered metaphysically possible. Alternatively, without invoking exotic metaphysical beings, there might be strong metaphysical reasons to regard finitude as an intrinsic feature of time. To be clear, we have no such reason to offer, let alone any intention to defend the legitimacy of malevolent demons. What we claim is that the metaphysical view according to which time unfolds indefinitely into the future stands, at least *prima facie*, on a par with the hypothesis that time may come to an end. To the best of our knowledge, no compelling metaphysical argument favors one view over the other. Consequently, the possibility that time might terminate remains a live metaphysical option, one that deserves to be considered and formally modeled. And in fact, the situation is even more striking: not only is it just as metaphysically possible for time to end as it is for it not to end, but it is also possible for it to end at any moment. To clarify, the claim that time can come to an end means that the temporal sequence of moments could have a final element. But once we grant this, a natural question arises: which moments are eligible to be the last? One option is that not all of them are: some instants are necessarily followed by others. Alternatively, every instant is a potential doomsday. We are inclined to

[12] See, for example, Kment (2021, §2).

favor the latter. The reason for this lies in the lack of metaphysical justification for the asymmetry that would arise among different moments, some of which would qualify as potential doomsdays, while others would not. Nevertheless, it remains possible to adopt a more moderate account—though still one of radical indeterminacy—in which not every moment is a potential doomsday. For the formal construction of structures that realize these two conceptions of time under radical indeterminacy, we now turn to the next section.

3 Incorporating Dead Branches

This section aims to provide a formal construction of trees with dead branches. As mentioned at the end of §2, there are essentially two alternatives we intend to consider. The first option corresponds to trees where every moment is a potential doomsday; the second, to trees where some moments are potential doomsdays, but not all. To keep things simple, in both approaches we assume that time has no beginning, thereby imposing BU on our structures. Though, for reasons similar to those offered in support of including forward-bounded branches, one might instead allow for branches that are also bounded in the past. Note, however, that the situation is not entirely symmetrical. If backward-branching is excluded, one is effectively modeling a closed past, so that the beginning of time is no longer contingent, but necessary: all histories will have one and the same initial moment. The relevant considerations are therefore different and call for a separate analysis.[13]

Let us begin by examining the first of the two alternatives. Suppose that at moment t, doomsday is possible but not necessary. In other words, t could be the last moment, but it is not necessarily so. How can this situation be represented within BTS? Arguably, we are required to include a history h that ends at t and another history h' that is a $<$-extension of h, in which moments after t exist. Consequently, since $h \subset h'$, the usual requirement of maximality that is typically considered definitional of histories in temporal logic would have to be abandoned (cf. Andreoletti 2022 for a similar view). To be more precise, we have to retain what we can call Backward-Maximality while dropping Forward-Maximality. Intuitively, a subset of $\mathcal{T}$ is backward-maximal when it cannot be extended to the past with respect to $<$, while it is forward-maximal when it cannot be extended to the future. Formally, given $h \subseteq \mathcal{T}$ that is linear with respect to $<$:

- h is backward-maximal iff for any other linear $h' \subseteq \mathcal{T}$ such that $h \subseteq h'$, there is no $t' \in h'$ such that for every $t \in h$, $t' < t$;
- h is forward-maximal iff for any other linear $h' \subseteq \mathcal{T}$ such that $h \subseteq h'$, there is no $t' \in h'$ such that for every $t \in h$, $t < t'$.

Accordingly, while preserving Linearity, we take histories not to be extendable into the past (i.e., they are backward-maximal), but extendable into the future (i.e., they do not have to be forward-maximal). This move forces every moment to be a potential doomsday, so that even if $\mathcal{T}$ is upper-unbounded, we will still

[13] In any case, adjusting the frame to constrain the past to have a beginning is straightforward, and the semantic consequences are fairly immediate.

have histories that come to an end—and that may end at different time instants. We denote such a tree structure by $\mathcal{T}_{DB}$. Since we have relaxed the maximality condition for the subsets of $\mathcal{T}_{DB}$ that qualify as histories, we adopt a history-based approach,[14] and make the set $\mathcal{H}$ of histories explicit in $\mathcal{T}_{DB}$. Then, we get $\mathcal{T}_{DB} = \langle T, <, \mathcal{H} \rangle$, where $\mathcal{H} = \{h : h \subseteq \mathcal{T} \wedge h \text{ is linear} \wedge h \text{ is backward-maximal}\}$, and $\mathcal{M}_{DB} = \langle T, <, \mathcal{H}, V \rangle$. An example of such a tree structure is shown in Figure 2.

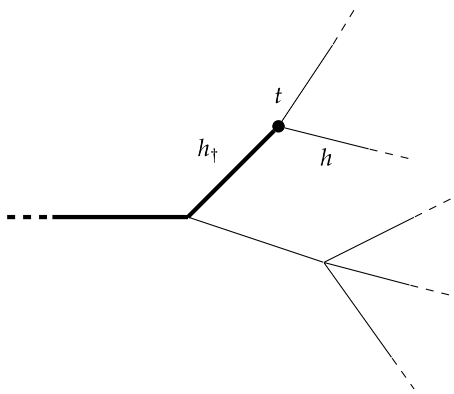

Figure 2: An example of a $\mathcal{T}_{DB}$ structure, that is, a structure which, in addition to satisfying the conditions of standard tree models, is defined by histories that are linear and backward-maximal, but not required to satisfy forward-maximality. This makes every moment of $\mathcal{T}_{DB}$ a potential doomsday. In the figure, this feature is illustrated for moment t, which is the doomsday of history $h_\dagger$ (the thick line), while being an ordinary (i.e., non-final) moment of history h. The same applies analogously to every moment of $\mathcal{T}_{DB}$.

The approach we have just discussed ensures that every moment is a potential doomsday, with significant semantic implications that will be examined in the following section. However, as noted in the introduction, there is a wide variety of trees corresponding to different ways of modeling time. In line with this, we now consider a second, more moderate proposal. We can set aside metaphysical considerations and incorporate the end of time in the most coherent way possible, as it is understood strictly within physics. Instead of granting that doomsday is possible at every moment, one might allow for the existence of moments that are necessarily not the last, while others may be. While from a metaphysical standpoint the distinction between moments that are possible doomsdays and those that are not does not appear to be justified, once we take into account only the physical laws and the state of the world at a given moment, it becomes credible that there are moments at which it is physically determined that the world will continue into the next instant, and others at which it is possible for time to come to an end. In other words, if the end of time is a physical possibility—or even a necessity—at certain moments, at others the state of the world and the physical laws guarantee the continuation

[14] On this point we essentially agree with Andreoletti (2022)

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of time. After all, our best cosmological models rule out the possibility that the moment at which we finish writing this sentence could be the last—luckily.

To design a tree structure that reflects this perspective, one option is to adopt what Sabbadin and Zanardo (2003) and Zanardo (2006a,b) call the *geometrical approach* to temporal logic. Here, both moments and histories are primitive entities, with no set-theoretical and ontological dependency of the latter on the former, in the same way as points and lines are regarded in many presentations of geometry. We refer to the structure that realizes this approach as $\mathcal{T}_G$. Then, we get $\mathcal{T}_G = \langle T, <, \mathcal{H}, E\rangle$, where $\mathcal{H}$ is a set of histories and E is a binary relation between moments $t \in T$ and histories $h \in \mathcal{H}$. The intended meaning of $E(t, h)$ is that h passes through t or t is encountered in h, thus providing a complete characterization of the relation between moments and histories. As usual, $<$ is a strict partial order and satisfies BL and HC.

Further constraints concerning the interactions between moments and histories and between $<$ and E (Zanardo, 2006b, p. 395) are the following:

(A1) $\forall t\, \exists h\, E(t, h) \wedge \forall h\, \exists t\, E(t, h)$

(A2) $\forall h\, \forall t_1 \forall t_2\, (E(t_1, h) \wedge E(t_2, h) \rightarrow (t_1 < t_2 \vee t_2 < t_1 \vee t_1 = t_2))$

(A3) $\forall h\, \forall t_1\, (E(t_1, h) \rightarrow \forall t_2\, (t_2 < t_1 \rightarrow E(t_2, h)))$

A1 ensures that no history is 'empty' of moments and that no moment is 'isolated' from all histories. That is, all histories pass through at least one moment and for every moment, there exists at least one history passing through it. A2 guarantees that any two moments through which a given history passes are linearly ordered by $<$. Finally, A3 ensures that if a history passes through a given moment, then the same history passes through all moments preceding it. Crucially, a key feature of this approach is that histories are not maximal sets of moments, and one history can be an extension of another. As a result, this approach is flexible enough to allow for the representation of both moments that are potential doomsdays and moments that are necessarily not—i.e., moments that are followed by at least one successor in every history branching from them.[15] All that is required is to define a suitable relation E. More precisely, for t_1 to be a potential doomsday, E must fulfill the following condition:

(A4) $\exists h \exists h_\dagger (E(t_1, h) \wedge E(t_1, h_\dagger) \wedge \exists t_2 (t_1 < t_2 \wedge E(t_2, h)) \wedge \neg \exists t_3 (t_1 <$
$< t_3 \wedge E(t_3, h_\dagger)))$

A4 states that a moment is a potential doomsday if it lies on at least two histories, one of which passes through some moment after it, while the other does not. On the other hand, for t_1 to be necessarily not a doomsday, E must fulfill the following condition:

[15]It should be noted that, within this approach, the previous one can be reproduced if the existence of as many histories as there are chains of moments is postulated. A third solution is to treat histories as primitive and define moments based on histories, following the approach of Sabbadin and Zanardo (2003); Zanardo (2006a). However, we will not consider this approach here because it has an important limitation. Since moments are defined as sets of histories, there cannot exist two distinct moments through which exactly the same histories pass. In other words, from each moment, at least two distinct histories must diverge (more technically: each moment must be the maximal element of the intersection of at least two histories). It follows that the tree described by the topological approach is *totally branching*. This is a consequence we would like to avoid.

(A5) $\forall h(E(t_1, h) \rightarrow \exists t_2(E(t_2, h) \wedge t_1 < t_2))$

That is, for a moment to be necessarily not a doomsday, A5 requires that all histories passing through it extend to at least one later moment. Here, we are not concerned with defining a specific relation E by determining which moments should satisfy one condition or the other. Rather, it suffices for our purposes to have shown that such structures can indeed be developed. This concludes the present section and allows us to examine how $\mathcal{T}_{DB}$ and $\mathcal{T}_G$ affect two of the most well-known BTSs.

4 The Impact of Dead Branches on the Ockhamist and Peircean BTSs

In this section, we offer a preliminary analysis of the consequences that the introduction of dead branches entails for Ockhamism and Peirceanism. For each of these, we begin by briefly outlining its key elements—beyond those already covered in the Introduction—and then proceed to prove and discuss a series of results.

4.1 The consequences of $\mathcal{T}_{DB}$ for Ockhamist semantics

To begin with, let $\mathcal{L}$ be the language defined as follows, where *Prop* is the set of propositional variables:

$$\phi := p \in Prop \mid \neg\phi \mid (\phi \wedge \phi) \mid P\phi \mid F\phi \mid \Diamond\phi$$

Based on the tree structure $\mathcal{T}$ discussed in §1, we now define a Ockhamist model $\mathcal{M}_{Ockham} = \langle T, <, V_{Ockham} \rangle$ where $V_{Ockham} := T \times \mathcal{H} \times Prop \mapsto \{1, 0\}$. The assignment of truth values to an arbitrary formula ϕ at a given moment-history pair (t, h) is defined inductively by the following semantic clauses:

$\mathcal{M}_{Ockham}, (t, h) \models p$ iff for $p \in Prop$, $V_{Ockham}(t, h, p) = 1$

$\mathcal{M}_{Ockham}, (t, h) \models \neg\phi$ iff $\mathcal{M}_{Ockham}, (t, h) \not\models \phi$

$\mathcal{M}_{Ockham}, (t, h) \models \phi \wedge \psi$ iff $\mathcal{M}_{Ockham}, (t, h) \models \phi$ and $\mathcal{M}_{Ockham}, (t, h) \models \psi$

$\mathcal{M}_{Ockham}, (t, h) \models P\phi$ iff $\mathcal{M}_{Ockham}, (t', h) \models \phi$ for some $t' \in h$ such that $t' < t$

$\mathcal{M}_{Ockham}, (t, h) \models F\phi$ iff $\mathcal{M}_{Ockham}, (t', h) \models \phi$ for some $t' \in h$ such that $t < t'$

$\mathcal{M}_{Ockham}, (t, h) \models \Diamond\phi$ iff $\mathcal{M}_{Ockham}, (t, h') \models \phi$ for some $h' \in \mathcal{H}(t)$[16]

Then, we define the usual dual operators as follows:

$H\phi \equiv \neg P \neg\phi$, so that $\mathcal{M}_{Ockham}, (t, h) \models H\phi$ iff $\mathcal{M}_{Ockham}, (t', h) \models \phi$ for all $t' \in h$ such that $t' < t$

[16] $\Diamond$ is the *historical possibility* operator, to be read as "it is still historically possible that." Its dual, i.e. $\Box$, is the *historical necessity* operator, to be read as "it is settled that" (Cf. Spolaore and Zanardo (2025, §3)).

$G\phi \equiv \neg F\neg\phi$, so that $\mathcal{M}_{Ockham}, (t,h) \models G\phi$ iff $\mathcal{M}_{Ockham}, (t',h) \models \phi$ for all $t' \in h$ such that $t < t'$

$\Box\phi \equiv \neg\Diamond\neg\phi$, so that $\mathcal{M}_{Ockham}, (t,h) \models \Box\phi$ iff $\mathcal{M}_{Ockham}, (t,h') \models \phi$ for all $h' \in \mathcal{H}(t)$

Finally:

(*validity in the model*) $\mathcal{M}_{Ockham} \models \phi$ iff $\mathcal{M}_{Ockham}, (t,h) \models \phi$ for every (t,h) in $\mathcal{M}_{Ockham}$

(*validity in the frame*) $\mathcal{T} \models \phi$ iff $\mathcal{M}_{Ockham} \models \phi$ for every model $\mathcal{M}_{Ockham}$ of $\mathcal{T}$

(*logical consequence in the frame*) $\Sigma \models_{\mathcal{T}} \phi$ iff $\mathcal{M}_{Ockham} \models \phi$ for every model $\mathcal{M}_{Ockham}$ of $\mathcal{T}$ such that for every $\psi \in \Sigma$, $\mathcal{M}_{Ockham} \models \psi$

In the case where $\mathcal{T}$ satisfies both BU and FU, the results yielded by the Ockhamist BTS are well known.[17] Here, we aim to explore what happens when we work with $\mathcal{T}_{DB}$, that is, we want to draw some immediate consequences that the tree structure with dead branches yields when combined with the Ockhamist clauses for tense operators.

First, the (F, G)-free fragment of Ockhamist BTS is not affected by the presence of dead branches. This is because the structure of histories is irrelevant for the clauses governing the connectives, and each moment has a linear, unbounded past in both $\mathcal{T}$ and $\mathcal{T}_{DB}$. Next, recall that the following important fact, which we call Fact_0, holds in $\mathcal{T}_{DB}$:

Fact_0) for every moment t, there is always a history in $\mathcal{H}(t)$ for which t is its upper bound—i.e., its doomsday. From now on, we denote by $h_\dagger$ such a history for every given t.

Prima facie, this may seem to have little impact on the tense operators, since they do not involve quantification over histories. However, dead branches do have important consequences, the most relevant of which are outlined below:

Fact_1) $\mathcal{T}_{DB} \not\models F\phi \vee F\neg\phi$ (NO LFEM)[18]

Given any moment t, if we consider the history $h_\dagger$ that ends at t, then $\mathcal{M}_{Ockham}, (t,h_\dagger) \not\models F\phi$. Thus, $\mathcal{M}_{Ockham}, (t,h_\dagger) \not\models F\neg\phi$. Consequently, $\mathcal{M}_{Ockham}, (t,h_\dagger) \not\models F\phi \vee F\neg\phi$, and therefore $\mathcal{T}_{DB} \not\models F\phi \vee F\neg\phi$. This marks a substantial departure from Ockhamist semantics without dead branches, where the Law of Future Excluded Middle (LFEM) holds.

Fact_2) $\mathcal{T}_{DB} \not\models P\phi \to \Box P\phi$ (NO Past Necessitation, PN)

Recall that $\phi \to \psi \equiv \neg\phi \vee \psi$. As in the standard Ockhamist BTS, Fact_2 can be proved by observing that $PFp \to \Box PFp$ will not hold in general in an Ockhamist branching time model—not even when we are working within $\mathcal{T}_{DB}$.

[17] See e.g. Goranko (2023, Ch. 7).

[18] Disjunction is defined in the usual way in terms of conjunction and negation.

Fact$_3$) $\mathcal{T}_{DB} \models \neg\Box F\phi$

It follows from Fact$_0$ and the fact that, for every moment t, if we consider the history $h_\dagger$ that ends at t, then $\mathcal{M}_{Ockham}, (t, h_\dagger) \not\models F\phi$. Moreover, from $\mathcal{T}_{DB} \models \neg\Box F\phi$, it also follows that $\mathcal{T}_{DB} \not\models \Box F\phi$. Fact$_3$ captures an important aspect of radical indeterminacy, namely that there are no necessary future facts. All future truths, tautologies included, are future contigents.

Fact$_4$) $\mathcal{T}_{DB} \not\models G\phi \to F\phi$

Fact$_0$ guarantees that for every t there is an history $h_\dagger \in \mathcal{H}(t)$ that ends at t, so that $\mathcal{M}_{Ockham}, (t, h_\dagger) \not\models F\phi$. But $\mathcal{M}_{Ockham}, (t, h_\dagger) \models G\phi$, since it is vacuously true that $\mathcal{M}_{Ockham}, (t', h_\dagger) \models \phi$ for all $t' \in h_\dagger$ such that $t < t'$. Therefore, $\mathcal{M}_{Ockham}, (t, h_\dagger) \not\models G\phi \to F\phi$, hence $\mathcal{T}_{DB} \not\models G\phi \to F\phi$. In turn, Fact$_4$ entails that $\mathcal{T}_{DB} \not\models \Box(G\phi \to F\phi)$.

Fact$_5$) $\mathcal{T}_{DB} \not\models \Box G\phi \to \Box F\phi$

Note that a model $\mathcal{M}_{Ockham}$ such that $\mathcal{M}_{Ockham}, (t, h) \models \Box G\phi$ for some t and h is possible. But from Fact$_3$, $\mathcal{M}_{Ockham}, (t, h) \not\models \Box F\phi$. Therefore, $\mathcal{M}_{Ockham}, (t, h) \not\models \Box G\phi \to \Box F\phi$, and thus $\mathcal{T}_{DB} \not\models \Box G\phi \to \Box F\phi$.

Arguably, the most significant result we have presented is Fact$_3$, as it entails the falsity of all future (alethic) necessities—e.g., "Tomorrow $2 + 2 = 4$."[19] While some may find it somewhat surprising, we do not consider it as counterintuitive. Under radical indeterminacy, where every moment is a potential doomsday, it might not be all that strange for every future statement to come out contingent, since there is no guarantee that a future in which the relevant fact occurs will actually come to be. And in the absence of such a future, it seems reasonable to regard the statement as false.[20] On the other hand, we take Fact$_4$ to be strongly counterintuitive. After all, $G\phi \to F\phi$ represents the very intuitive temporal analogue of the modal principle $\Box\phi \to \Diamond\phi$ (i.e., metaphysical necessity implies possibility), which is widely accepted. The failure of this principle follows directly from the fact that any $G\phi$ turns out to be vacuously true when evaluated at any moment t along the history that ends at t. Yet this, too, runs counter to our intuition: at the final moment, it seems that no proposition will ever be true, rather than that each proposition will always be true. Furthermore, notice that $G\bot$ is valid at every final moment of any dead history. Given that this feature is the source of the undesirable Facts$_{4,5}$, it helps motivate the development of the alternative semantics proposed in §5.

4.2 The consequences of $\mathcal{T}_{DB}$ for Peircean semantics

Let $\mathcal{L}$ be the language now defined as:

$$\phi := p \in Prop \mid \neg\phi \mid (\phi \wedge \phi) \mid P\phi \mid F\phi \mid G\phi$$

[19] We are assuming that all necessary propositions can fall within the scope of a temporal operator. However, one might object that this is not the case: at least some necessary propositions are atemporal and therefore cannot fall within the scope of temporal operators. If "2+2=4" is atemporal, then "Tomorrow 2+2=4" is ill-formed. We do not take a stance on this issue and simply qualify our thesis as follows: Fact$_3$ implies that, given a necessary proposition p that can fall within the scope of temporal operators, "Tomorrow p" is false.

[20] A perhaps even more intuitive alternative is to treat all future statements as indeterminate within the framework of radical indeterminism. More on that in §5.

Based on $\mathcal{T}$, we now define a Peircean model $\mathcal{M}_{Peirce} = \langle T, <, V_{Peirce}\rangle$ where $V_{Peirce} := Prop \times T \mapsto \{1, 0\}$. The assignment of truth values at a given moment t is extended to an arbitrary formula ϕ inductively by the following semantic clauses:

$\mathcal{M}_{Peirce}, t \models p$ iff for $p \in Prop$, $V_{Peirce}(t, p) = 1$

$\mathcal{M}_{Peirce}, t \models \neg\phi$ iff $\mathcal{M}_{Peirce}, t \not\models \phi$

$\mathcal{M}_{Peirce}, t \models \phi \wedge \psi$ iff $\mathcal{M}_{Peirce}, t \models \phi$ and $\mathcal{M}_{Peirce}, t \models \psi$

$\mathcal{M}_{Peirce}, t \models P\phi$ iff $\mathcal{M}_{Peirce}, t' \models \phi$ for some t' such that $t' < t$

$\mathcal{M}_{Peirce}, t \models F\phi$ iff for all $h \in \mathcal{H}(t)$, there is some $t' \in h$ such that $t < t'$ and $\mathcal{M}_{Peirce}, t' \models \phi$

$\mathcal{M}_{Peirce}, t \models G\phi$ iff for all $h \in \mathcal{H}(t)$, for every $t' \in h$ such that $t < t'$, $\mathcal{M}_{Peirce}, t' \models \phi$

Then, we can define the usual dual operators as follows:[21]

$H\phi \equiv \neg P\neg\phi$, so that $\mathcal{M}_{Peirce}, t \models H\phi$ iff $\mathcal{M}_{Peirce}, t' \models \phi$ for all t' such that $t' < t$

$g\phi \equiv \neg F\neg\phi$, so that $\mathcal{M}_{Peirce}, t \models g\phi$ iff for some $h \in \mathcal{H}(t)$, for every $t' \in h$ such that $t < t'$, $\mathcal{M}_{Peirce}, t' \models \phi$

$f\phi \equiv \neg G\neg\phi$, so that $\mathcal{M}_{Peirce}, t \models f\phi$ iff for some $h \in \mathcal{H}(t)$, there is some $t' \in h$ such that $t < t'$ and $\mathcal{M}_{Peirce}, t' \models \phi$

Finally:

(*validity in the model*) $\mathcal{M}_{Peirce} \models \phi$ iff $\mathcal{M}_{Peirce}, t \models \phi$ for every t in $\mathcal{M}_{Peirce}$

(*validity in the frame*) $\mathcal{T} \models \phi$ iff $\mathcal{M}_{Peirce} \models \phi$ for every model $\mathcal{M}_{Peirce}$ of $\mathcal{T}$

(*logical consequence in the frame*) $\Sigma \models_{\mathcal{T}} \phi$ iff $\mathcal{M}_{Peirce} \models \phi$ for every model $\mathcal{M}_{Peirce}$ of $\mathcal{T}$ such that for every $\psi \in \Sigma$, $\mathcal{M}_{Peirce} \models \psi$

In the case where $\mathcal{T}$ satisfies both BU and FU, the results (i.e., the valid formulas) yielded by the Peircean BTS are well known.[22] Again, what we aim to do in this subsection is to explore what happens when we work with $\mathcal{T}_{DB}$.

First, note again that the (F, G)-free fragment of Peircean BTS with dead branches is equivalent to the (F, G)-free fragment of Peircean BTS without dead branches. Fact_0 still holds. This leads to the following further facts:

Fact_6) $\mathcal{T}_{DB} \models \neg F\phi$

It follows from Fact_0, which makes $\mathcal{M}_{Peirce}, t \not\models F\phi$ for every t in $\mathcal{M}_{Peirce}$ and for every $\mathcal{M}_{Peirce}$. This is a key feature of this semantics in the presence of dead branches, and marks a substantial difference from the standard Peircean BTS. While the latter semantics makes

[21] Recall that in Peircean BTS, the strong tense operators F and G are not duals of each other, but rather they are the duals of their respective weak tense operators, g and f.

[22] See e.g. Goranko (2023, Ch. 6).

all future contingents false but allows for true future statements, the former treats every future statement as false. There is simply no truth about the future! Moreover, this shows that $\neg F\phi \not\equiv F\neg\phi$ (as was the case in the standard version).

Fact$_7$) $\mathcal{T}_{DB} \not\models F\phi \vee F\neg\phi$ (NO LFEM)

It follows from Fact$_6$. In other words, LFEM is not valid, just as in standard Peircean semantics.

Fact$_8$) $\mathcal{T}_{DB} \models g\phi$

It follows from $g\phi \equiv \neg F\neg\phi$ and Fact$_6$. Since $g\phi$ holds for every ϕ, the operator g becomes entirely trivial, to the extent that, in particular, $\mathcal{T}_{DB} \models g(\phi \wedge \neg\phi)$.

Fact$_9$) $\mathcal{T}_{DB} \not\models G\phi \rightarrow F\phi$

Fact$_0$ ensures that for any t there is a history $h_\dagger \in \mathcal{H}(t)$ that ends at t. On the one hand, it is vacuously true that for every $t' \in h_\dagger$ such that $t < t'$, $\mathcal{M}_{Peirce}, t' \models \phi$. This makes it possible to construct a model $\mathcal{M}_{Peirce}$ such that $\mathcal{M}_{Peirce}, t \models G\phi$. On the other hand, it is false that there is some $t' \in h_\dagger$ such that $t < t'$ and $\mathcal{M}_{Peirce}, t' \models \phi$ (cf. Fact$_6$), so that $\mathcal{M}_{Peirce}, t \not\models F\phi$. Therefore, $\mathcal{M}_{Peirce}, t \not\models G\phi \rightarrow F\phi$, hence $\mathcal{T}_{DB} \not\models G\phi \rightarrow F\phi$. This marks yet another major difference—as well as a highly counterintuitive result—compared to Peircean semantics without dead branches. Fact$_9$ is the Peircean counterpart of Fact$_5$ in the Ockhamist semantics.

The impact of $\mathcal{T}_{DB}$ on Peircean semantics parallels that on Ockhamist semantics and similar remarks to those made earlier apply here as well. Again, we take Fact$_9$ to be strongly counterintuitive, which motivates appropriate adjustments.

4.3 The consequences of $\mathcal{T}_G$ for Ockhamist and Peircean semantics

In the geometrical framework, not every t is necessarily a final moment of some history—i.e., a potential doomsday. That is, Fact$_0$ does not obtain. Since several of the facts we proved so far depend on it, they too fail to hold. Nonetheless, some of them—and in fact rather significant ones—still obtain.

In particular, with regard to the Ockhamist framework, the following interesting facts still obtain:

Fact$'_1$) $\mathcal{T}_G \not\models F\phi \vee F\neg\phi$ (NO LFEM)

Let us consider the final moment of a dead history. At that moment, neither $F\phi$ nor $F\neg\phi$ holds, and consequently, $F\phi \vee F\neg\phi$ does not hold either. It follows that $\mathcal{T}_G \not\models F\phi \vee F\neg\phi$.

Fact$'_4$) $\mathcal{T}_G \not\models G\phi \rightarrow F\phi$

Again, let us consider the final moment of a dead history. At that moment, $G\phi$ is vacuously true, whereas $F\phi$ is false because there is no later moment in that history. Therefore, $G\phi \rightarrow F\phi$ is false at that moment. As a consequence, $\mathcal{T}_G \not\models G\phi \rightarrow F\phi$.

With regard to the Peircean framework, the following fact still obtains:

Fact$'_{11}$) $\mathcal{T}_G \not\models G\phi \rightarrow F\phi$

Suppose that t is the terminal moment of a history $h_\dagger$. On the one hand, it is vacuously true that for every history h such that $h \in \mathcal{H}(t)$ and for every moment t' such that $E(t', h)$ and such that $t < t'$, $\mathcal{M}_{Peirce}, t' \models \phi$. On the other hand, it is false that there is some $E(t', h_\dagger)$ such that $t < t'$ and $\mathcal{M}_{Peirce}, t' \models \phi$ and, hence, $F\phi$ is false at t. Therefore, $\mathcal{M}_{Peirce}, t \not\models G\phi \rightarrow F\phi$, hence $\mathcal{T}_G \not\models G\phi \rightarrow F\phi$.

Once again, these results suggest the possibility of changing the semantics to avoid them.

Now, suppose that $h_\dagger$ is a dead history and that t is its final moment. So, we have the following results in $\mathcal{T}_G$:

$$\mathcal{M}_{Ockham}, (t, h_\dagger) \models \neg F\phi$$

$$\mathcal{M}_{Ockham}, (t, h_\dagger) \models \neg\Box F\phi$$

However, these results do not obtain at every moment. Indeed, the geometric framework allows for the existence of moments that are not the final moment of any history. It is therefore possible for there to be moments such that, for every history passing through them, there exists a subsequent moment at which a given proposition is true. Therefore, contrary to what happens in $\mathcal{T}_{DB}$ (see Fact$_3$), $\neg\Box F\phi$ is not valid in $\mathcal{T}_G$.

This also has an impact on the Peircean framework. Again, suppose that $h_\dagger$ is a dead history and that t is its final moment. Then:

$$\mathcal{M}_{Peirce}, t \models \neg F\phi$$

$$\mathcal{M}_{Peirce}, t \models g\phi$$

But once again, the existence of moments that are not the last of any history ensures that these results do not hold at every instant in the structure. Therefore, contrary to what happens in $\mathcal{T}_{DB}$, $\neg F\phi$ and $g\phi$ are not valid in $\mathcal{T}_G$.

5 Proposals for emending the semantics

As shown in §4, the Ockamist and Peircean semantic clauses yield some implausible results on $\mathcal{T}_{DB}$ and $\mathcal{T}_G$. To avoid such consequences, in this section we propose a new semantics based on the Ockhamist approach that is specifically designed for tree structures with dead branches. The main aim of this semantics is to prevent $G\phi$ from being vacuously true at the terminal instant.

Our semantics makes use of the notion of *interval of available instants*. Briefly, the idea is as follows: given a moment in time (t) and a history (h), let I_t denote the set of future instants that exist relative to t. Since, at this stage, we shall not consider alternative histories, they may be disregarded. Accordingly, we shall have $I_t = \{t' \mid t' > t\}$. The set I_t may be infinite —- in the case of histories that do not come to an end –finite, in the case of histories that will eventually terminate, or even empty, in the case where we find ourselves at the final instant of time.

Truth clauses for formula containing future operators are the followings:

$$\mathcal{M}_{Ockham}, (t,h) \models G\varphi \text{ iff } I_t \neq \varnothing \text{ and } \forall t' \in I_t, t'/h \models \varphi$$

$$\mathcal{M}_{Ockham}, (t,h) \models F\varphi \text{ iff } I_t \neq \varnothing \text{ and } \exists t' \in I_t, t'/h \models \varphi$$

As a result, $\mathcal{T}_{DB} \models G\phi \rightarrow F\phi$ and $\mathcal{T}_G \models G\phi \rightarrow F\phi$. Moreover, the contradiction is no longer always true "in the future" of the final instant. One aspect that requires clarification with respect to these clauses concerns negation. Since the truth clause is a conjunction, the failure of truth ($\not\models$) is compatible with the negation of the first conjunct, the negation of the second conjunct, or the negation of both. Two distinct strategies may be adopted in this context. The first does not allow for truth-value gaps and declares $G\phi$ and $F\phi$ to be false if either of the conjuncts in their respective truth clauses is false. It follows that $G\phi$ and $F\phi$ are both false at the doomsday. Since this strategy preserves bivalence, the failure of truth is equivalent to falsity and to the truth of the negation.

$$\mathcal{M}_{Ockham}, (t,h) \not\models G\varphi \text{ iff either } I_t = \varnothing \text{ or } \exists t' \in I_t, t'/h \not\models \varphi$$

$$\mathcal{M}_{Ockham}, (t,h) \models \neg G\varphi \text{ iff either } I_t = \varnothing \text{ or } \exists t' \in I_t, t'/h \not\models \varphi$$

$$\mathcal{M}_{Ockham}, (t,h) \not\models F\varphi \text{ iff either } I_t \neq \varnothing \text{ or } \forall t' \in I_t, t'/h \not\models \varphi$$

$$\mathcal{M}_{Ockham}, (t,h) \models \neg F\varphi \text{ iff either } I_t \neq \varnothing \text{ or } \forall t' \in I_t, t'/h \not\models \varphi$$

However, it should be noted that F and G are no longer dual and interdefinable within this strategy. In particular, $G\phi$ will not be equivalent to $\neg F \neg \phi$ at the final instants of any dead history.

The second strategy arises from the observation that it makes little sense to evaluate a proposition at a moment that does not exist. In other words, the existence of an evaluation instant appears to be a precondition for assessing whether a proposition is true or false at that instant. The evaluation function takes, among other things, moments as its arguments; however, if a moment does not exist, then one of the arguments of the function is lacking. If an argument is missing, the value of the function cannot be computed. In such a case, the function yields no value. The existence of the instant can thus be treated as a presupposition, without which propositions cannot be judged either true or false. Consequently, there will be gaps in truth values in cases where the evaluation moment does not exist.

To implement this strategy, we shall assume that the truth conditions for $F\phi$ and $G\phi$ are partial functions from times and histories to truth values. $F\phi$ and $G\phi$ are true or false at a moment and a history only if that moment is not the final moment of that history. If it is, then both formulas have an indeterminate truth value. Since this amendment introduces gaps in truth values, the notion of "not being a model for a given proposition" is no longer equivalent to "being a model for the negation of that proposition." We therefore introduce the symbol $\dashv$, which is to be read as "being a countermodel of."

We shall therefore say that $G\varphi$ and $F\varphi$ are *untrue* if and only if there does not exist the interval of instants over which the sentences in question can be interpreted:

$$\mathcal{M}_{Ockham}, (t,h) \not\models G\varphi \text{ iff } I_t = \varnothing$$

$$\mathcal{M}_{Ockham}, (t,h) \not\models F\varphi \text{ iff } I_t = \varnothing$$

Conversely, we shall say that $G\varphi$ is *false* if and only if there exists an interval of available future instants, but within it at least one instant renders φ false:

$$\mathcal{M}_{Ockham}, (t, h) \dashv G\varphi \text{ iff } I_t \neq \varnothing \text{ and } \exists t' \in I_t, t'/h \models \neg\varphi$$

Similarly, $F\varphi$ is false if and only if there exists an interval of available future instants, but within all instants render φ false:

$$\mathcal{M}_{Ockham}, (t, h) \dashv F\varphi \text{ iff } I_t \neq \varnothing \text{ and } \forall t' \in I_t, t'/h \models \neg\varphi$$

Notice that within this strategy the negation of a proposition is true when the proposition is false:

$$\mathcal{M}_{Ockham}, (t, h) \models \neg\phi \text{ iff } \mathcal{M}_{Ockham}, (t, h) \dashv \phi$$

It follows that the negation of a gappy proposition is itself gappy. If t is a terminal moment of a history, then both the future tense propositions and their negations will lack a truth value with respect to that time and that history:

$$\mathcal{M}_{Ockham}, (t, h) \not\models \neg G\varphi \text{ iff } I_t = \varnothing$$

$$\mathcal{M}_{Ockham}, (t, h) \not\models \neg F\varphi \text{ iff } I_t = \varnothing$$

Under these clauses $G\phi \to F\phi$ is also a valid principle, and the contradiction is no longer always true in the future of the final instant. One of the advantages of this strategy is that F and G are once again dual and interdefinable with each other: $G\phi \equiv \neg F \neg \phi$.

With regard to the other Boolean logical operators, we may adopt a type of three-valued logic such as weak Kleene logic (Kleene (1952)). The underlying idea is that, in this logic, the indeterminacy of the truth value propagates, so to speak, in a compositional manner. Thus, for example, a conjunction in which one conjunct is indeterminate and the other true or false is itself indeterminate.

It is easy to see that in this kind of logic there are no tautologies. In particular, the shortcoming of such logics is that they do not validate fundamental logical principles—such as the principle of non-contradiction and the law of the excluded middle—when propositions are gappy. However, in this case, we do not consider that to be a defect. This strategy is based on the assumption that it is impossible to evaluate certain kinds of propositions at times that do not exist. Not even the principle of non-contradiction holds at a time that does not exist. Therefore, in this context, it is legitimate to continue to adopt such a logic. It should be noted, finally, that if a moment exists, then the *temporal free* propositions evaluated at that moment are never truth-valueless. In other words, as far as temporal free propositions are concerned, bivalence holds at all existing moments, and consequently, the logical principles of non-contradiction and the law of the excluded middle also hold at such moments.

6 Conclusions and Future Perspectives

In this paper, we have argued that branching time semantics can and should be extended to accommodate a more radical form of indeterminacy: not only is it open which course of events the future will take, but it is also open whether the future will continue to unfold at all. To capture this idea, we introduced temporal frames with dead branches and examined their implications for some of the most influential approaches to branching time semantics.

Our analysis has shown that incorporating dead branches has significant consequences for both Ockhamist and Peircean semantics. In particular, several principles that are valid in the standard setting break down once histories that come to an end are admitted. While some of these results may be seen as natural consequences of radical indeterminacy, others appear deeply counterintuitive. This motivates the need for an alternative framework. To this end, we have proposed a modified Ockhamist semantics that prevents vacuous truth at terminal instants and can be developed either within a bivalent setting or by adopting a gappy semantics.

The proposal we have outlined is only a first step. Much remains to be explored, both technically and philosophically. On the technical side, it would be worth investigating in greater detail the interaction between dead branches and alternative semantic frameworks—such as the supervaluationist approach—and clarifying the algebraic properties of the resulting logics. On the philosophical side, our account invites further reflection on the metaphysics of time, the nature of future contingents, and the ways in which cosmological scenarios of temporal finitude can inform formal semantics. Taken together, these considerations suggest that branching time semantics with dead branches offers a fertile ground for future work. By broadening the scope of what counts as an open future, it provides a powerful formal tool to represent the possibility that time itself may come to an end, and to reassess, from a new perspective, the conceptual landscape of indeterminism.

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