Prior's System Q and its Extensions

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Abstract

Recently, and as happens from time to time in New Zealand, a typescript of Prior's turned up. This one was in the personal collection of Oliver Sutherland. Prior typed it in November 1957 and used copies in his senior logic group, an informal research group at Canterbury University. A terse and relentlessly compressed couple of pages, it concerns Prior's system Q, which even towards the end of his life he was still describing as 'the true modal logic'. We analyse Prior's typescript and the issues underlying it, as well as providing an exposition of Q, and an examination of Łukasiewicz's objections to Q. The article also includes an interview with Prior's student Robert Bull concerning Q.

Keywords: Arthur Prior, Robert Bull, Jan Łukasiewicz, John Lemmon, Krister Segerberg, John Mackie, Time and Modality, system Q, temporal logic, modal logic, multi-valued logic, necessary existence, contingent existence, unstatable proposition, history of modal logic, Barcan formula, axiomatisation of modal logic, completeness proofs for modal logic.

1. Introduction

In 1957, or perhaps early 1958, Prior gave Nancy Sutherland a copy of his 1957 book *Time and Modality*, and also two pages of typewritten notes. Professor Ivan Sutherland, Nancy's not long deceased husband—he died in 1952—had been head of the department that Prior joined when appointed to a lectureship at Canterbury University College in 1946.[1] The book is inscribed 'With best wishes—ANP'. In due course, it and the notes were inherited by the Sutherland's son Oliver. Although Oliver was only a boy when Prior left New Zealand for the University of Manchester, in 1958, he remembers him well. 'Arthur had a nice grin', he says. 'He grinned a lot. Arthur was a very pleasant man, affable and friendly.'

In 2022, Martin Prior, Prior's son, visited Oliver. Oliver gave him the notes, and Martin gave a copy to us. Dated November 1957, the notes are titled 'Some Extensions to the System Q of "Time and Modality"'. Prior's Q is an important system in temporal and modal logic. He said in material he was working on at the end of his life:

> If neither S5 nor S2 is the true modal system, what is? I would stick by the answer which I gave in *Time and Modality*, namely that it is the system which I have called Q.
>
> (Prior & Fine 1977: 103)

Prior referred to Q frequently, and over the years developed it continuously. As late as 1967, he was still considering the difficult problem of developing a 'modification of tense logic analogous to Q' (Prior 1967: 155). Two classic papers on Q were published in the 1960s by Bull and Segerberg (Bull 1964; Segerberg 1967). More recent studies include Correia 1999, 2001, Akama & Nagata 2005, Akama, Nagata & Yamada 2008, and Badie 2021.

Q was first presented in Chapter V of *Time and Modality* (Prior 1957a: 41ff). The 1957 notes, which were composed after *Time and Modality* was published, describe Prior's further thinking on Q, and the treatment given in the notes is a significant development of the presentation in the book. These notes aid us in understanding how Prior transitioned from the matrix presentation of Q in *Time and Modality* to his later axiomatic formulation of the system, as summarised in, e.g., *Past, Present and*

[1] For additional information on Ivan and Nancy Sutherland, see Sutherland 2013.

Future (Prior 1967: 154ff) and *Papers on Time and Tense* (Prior 1968a: 148ff, 2003: 260ff).

Section 2 gives a short exposition of Prior's quirky system Q. Section 3 is an overview of the innovations in the 1957 notes. Section 4 presents the 1957 notes in full. Section 5 contains Prior's brief later addendum to the notes, from his Nachlass in the Bodleian Library. Section 6 reviews Łukasiewicz's objections to Q. Section 7 records an interview with Prior's student Robert Bull. Bull axiomatized Q and gave the first completeness proof for Prior's system (Bull 1964). In the interview, Bull disagrees with Łukasiewicz on some fundamental matters concerning Q. The concluding Section 8 outlines developments that followed in the wake of Prior's fertile 1957 notes.

2. The system Q

Prior explains in *Time and Modality* that his concerns about the Barcan formula play a key role in Q's motivation. In the opening chapters of the book, he constructs tense-logical analogues of the Lewis systems S4 and S5, and then, following a presentation of his definitive objection to the Barcan formula, he says: 'The dubiety of the Barcan formula is … transmissible to the entire structure of the tense-logic we have so far erected' (Prior 1957a: 27).

Prior's tense-logical future-tensed form of the Barcan formula, $CF\Sigma x\phi x\Sigma xF\phi x$ [$F\Sigma x\phi x \rightarrow \Sigma xF\phi x$] asserts: 'If it will be the case that something ϕ's, then there is something which will ϕ' (Prior 1957a: 29). To this he offers a counterexample: 'suppose that in fact someone will fly to the moon some day, but not anyone who now exists' (ibid.: 26, 29). 'The trouble' with $CF\Sigma x\phi x\Sigma xF\phi x$, he says, 'is clearly that the consequent asserts that something already existing will ϕ, while the antecedent does not commit us to that much' (ibid.: 29). He continues:

> [T]he only ground one can think of for assenting to it [the Barcan formula] would be a conviction that whatever is going to exist at some future time exists already. And if $CF\Sigma x\phi x\Sigma xF\phi x$ is laid down as expressing a logical law, i.e. as yielding with all concrete substitutions for its variables a statement which is true whenever it is made, it can only be justified by the assumption that whatever exists at any time exists at all times, i.e. the assumption that all real individuals are sempiternal.
>
> (ibid.)

Q, Prior says, is a tense logic that is not 'haunted by the myth that whatever exists at any time exists at all times' (ibid.: 48). To dispel this myth,

> we must think differently of the range of values for bound variables in this logic. 'There *is* an x which ϕ's' must be understood as true if we can now frame a true statement of the form '$x\phi$'s'; and 'There *will be* an x tomorrow which ϕ's', or 'It will be the case tomorrow that there is an x which ϕ's' must be understood as true if we will be able tomorrow to form a true statement of the form '$x\phi$'s', even if it be a statement which not only would not be true now but could not be framed now.
>
> (ibid.: 32)

In Q, what is statable tomorrow may not be stateable today—because, for instance, the statement may contain a logically proper name, and if not all individuals are sempiternal, it may be the case that the name's referent exists tomorrow, but not today. The idea that statability is time-dependent is core to Q. Prior said:

> [L]et us again begin by supposing that there are only two times, … today and yesterday. And we shall assume that today is the day on which we are using this system, so that its variables can stand for any statement which we can make today, whether we could have made it yesterday or not. The possibility that some of our statements could not have been made yesterday enlarges the possible values of our statements from four to six: (1) true at both times; (2) true today and unstatable yesterday; (3) true today and false yesterday; (4) false today but true yesterday; (5) false today and unstatable yesterday; and (6) false at both times.
>
> (ibid.: 41)

Axiomatic presentations of Q were initially elusive. In *Time and Modality* Prior had no axiomatisation of Q to offer. He said: '[A] many-valued system … will … be the most precise mathematical representation of [Q] that is possible until a complete set of postulates for it is found' (Prior 1957a: 40). The first provably complete axiomatisation of the system was devised much later, by Bull (Bull 1964). Prior gave simply a matrix formulation in *Time and Modality*, pursuing Łukasiewicz's methodology of using many-valued matrices to define logical systems. Prior noted early on in *Time and Modality* that this methodology would play an important role:

> [T]he treatment of modal logics as many-valued truth-functional systems, with the notion of a 'truth-value' enlarged a little, does provide us with an extremely fruitful instrument of inquiry.
>
> (ibid.: 7)

Prior's phrase 'enlarged a little' marks a significant departure from Łukasiewicz's own use of the method (a departure that first appeared in Prior 1952). Prior puts his own spin on Łukasiewicz's many-valued matrices, interpreting some of the values as, e.g., truth-at-a-time or truth-at-a-world. In the case of Q, the notion of a 'truth-value' is further enlarged to include statability-at-a-time.

Time and Modality contains tables for a range of Q's unary operators, *N* (negation), *M* (possibility), *L* (necessity), and *NMN* (in effect another, different, necessity operator); and there are further tables for the binary operators of conjunction, disjunction, and implication (Prior 1957a: 41-43). *Np*, *Mp* and *Lp* are all unstatable at a time if *p* is unstatable at that time, and similarly for binary combinations involving an unstatable *p*. Prior deems 1 and 2 the *designated* values, marking them '*': a formula is said to express a logical law just in case its value is either 1 or 2 in every possible case (ibid.: 41). The tables for the unary operators are:

p	*Np*	*Mp*	*Lp*	*NMNp*
*1	6	1	1	1
*2	5	2	5	2
3	4	1	6	6
4	3	1	6	6
5	2	5	5	5
6	1	6	6	6

As these tables show, the values of the two modalities *L* and *NMN* diverge when *p*'s value is 2 (true today and unstatable yesterday). In effect, *Lp* means 'true at both times' and *NMNp* means 'false at neither time'. When *p*'s value is 2, so is *NMNp*'s, but the value of *Lp* is the undesignated 5 (false today and unstatable yesterday)—and so, as Prior points out, there are no laws of the form *L*… in Q. The same is not true of *NMN*.

Prior explains that these '6-valued tables are in fact just the first step towards the matrix with an infinite number of elements which would give us the exact many-valued equivalent of the system Q' (ibid: 43). The elements of the matrix are infinite sequences of (any or all of) the numbers 1, 2 and 3, with the proviso that 2 never occupies the first place of a sequence:

> Intuitively we can think of these numbers as indicating whether the proposition is true [1], unstatable [2], or false [3] at each one of an infinity of times, the first place in the sequence representing the present time.
>
> (ibid.)

The designated sequences of the infinite-matrix system 'are all those which contain no 3's' (ibid: 44). The principle for handling compound propositions is straightforward: all compounds are assigned 2 whenever any component is assigned 2, and are treated classically with regard to assignments other than 2. Prior's matrix formulation of Q, although not as illuminating as an axiomatisation, does showcase the power and utility of Łukasiewicz's matrix method.

In the infinite-matrix semantics, *L* means 'always true' and *NMN* means 'never false' (Prior 1969: 337). Q's conspicuous distinctiveness is by no means limited to quantificational formulae. Two examples of non-theses at the modal propositional level are *CNMNpLp* [$-\Diamond-p\rightarrow\Box p$]—though its converse is a thesis—and *CMpMApq* [$\Diamond p\rightarrow\Diamond(p\vee q)$]. *CNMNpLp* fails when the infinite sequence for *p* contains only 1s and 2s (Prior 1957a: 45). Concerning *CMpMApq*, Prior says:

> [L]et *p* be 'Only God exists', and suppose this is possible, and let *q* be 'I don't exist'. … [T]he disjunction is unstatable unless I exist, and is therefore only statable if both parts of it, 'I don't exist' and 'Only God exists', are false. In this case, then, 'Possibly either *p* or *q*' is false although 'Possibly *p*' is true.
>
> (Prior 1957a: 49)

In the Sixties, Segerberg described Q as 'well-known but little understood' (Segerberg 1967: 54). It was Segerberg who plotted Q's position on the chart of more familiar systems. He introduced a bimodal axiomatic logic HS5, and an accompanying model theory in which a proposition may be *true*, *false* or *unstatable* at a world. His system's two differentiated necessity operators correspond to Prior's *Lp* and *NMNp*. In Segerberg's semantics, HS5—like its parent logic S5—requires an accessibility relation that is reflexive, transitive, and symmetrical. HS5, Segerberg says, 'bears a very close relationship to Q—one may even say that for practical purposes it *is* Q' (ibid.: 54).

The difference is that HS5, but not Q, contains an operator, here written ***s***, such that ***s****p* means '*p* is statable'.[2] Segerberg proved that 'Q agrees with HS5 on the set of well-formed formulas of Q' (ibid.: 68).

3. Innovations in the 1957 notes

Prior begins by adding a statability functor *S* to the syntax of Q. He says 'The idea of introducing the notion of statability directly into the symbolism is due to J.L.Mackie'.[3] Prior's *S* is not the same as Segerberg's later statability functor ***s***: *S* may be regarded as a necessitated form of ***s***. *Sp* means 'always statable', Prior says (Prior 1967: 154). *Lp* now means '*p* is always statable and never false', which is the same as *KSpNMNp* [*Sp & –◊–p*]. For propositions *p* of which *Sp* holds, the distinction between the modalities *Lp* and *NMNp* collapses.

Prior's leading concern in the 1957 notes was with axiomatisation, and he saw extending Q's language with *S* as the means to achieving an axiomatic system that has the original Q as a fragment. In the notes he gives axioms for an 'M-S logic' (i.e., a logic in *M* and *S*) that 'would appear to have Q for its M-L portion'. But at that stage he was simply not sure. We call this logic Q_{MS}, and we continue the story of Prior's efforts to axiomatize Q in Sections 5 and 8.

After presenting Q_{MS}, Prior goes on to explore four further extensions of Q, due to Lemmon. (Lemmon described Q as 'formally of great interest' (Lemmon 1958).) *Time and Modality* had presented Q as a

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[2] Segerberg wrote '+' where we write 's'.

[3] Prior's source for this idea may have been a four-page typescript in his Nachlass entitled 'An Imaginary Discussion on Time and Modality', and seemingly composed by Mackie. The discussion is between Mackie and Prior and, towards the end, the Mackie character suggests 'introducing an operator (say R) for "is unstatable"' (Mackie? n.d.: 4). (The typescript presumably dates from 1957, since it contains page references to the published version of Time and Modality, and Prior reported Mackie's suggestion in November of that year.)

‘modal logic for contingent beings’ (Prior 1957a: 50), and Lemmon objected that a logic for contingent beings should countenance no necessary beings—whereas Q countenances *both* contingent and necessary beings (Prior 1967: 155). Lemmon suggested that Q could be turned into a logic for (only) contingent beings by deleting from Q’s infinite matrix all sequences not containing at least one occurrence of 2. This is the system designated (ii) in Prior’s 1957 notes. In system (i), conversely, all sequences that do contain at least one occurrence of 2 are removed, giving a system that countenances only necessary beings. In system (iii), all sequences containing both 1 and 3 are removed; and in system (iv) the removed sequences are those containing both 1 and 3 but no occurrence of 2.

In the notes Prior uses his functor *S* in order to formulate axiomatisations of systems (i)–(iv). He gives four axioms that, if added singly to Q_{MS}, produce systems—we will call them S_i, S_{ii}, S_{iii}, and S_{iv}—whose M-L fragments are, respectively, (i), (ii), (iii), and (iv). He observes that system S_i contains Lewis’s S5; and concludes that S_i ‘exhibit[s] S5 clearly as the result of assuming that all beings are necessary beings’. His analyses of S_{ii}, S_{iii}, and S_{iv} are also illuminating.

Prior evidently sent the notes straight away to Lemmon in Oxford, who replied in December: ‘I’ve been very interested in this renewed activity on Q, but haven’t had time to absorb it yet—I hope to have a look at it next week more fully’ (Lemmon 1957). In a subsequent letter, Lemmon said: ‘It seemed to me that the introduction of ‘S’ for stateability was a great step forward formally here’ (Lemmon 1958).

4. Prior's 1957 notes

A.N. Prior, November, 1957.

SOME EXTENSIONS TO THE SYSTEM Q OF 'TIME AND MODALITY'

Introduce into tense-logic or modal logic a functor S such that

Sp = The statement that p, can always (or: in all possible circumstances) be made
= All individuals directly mentioned in the statement that p, always (or necessarily) exist
= For all t, either p at t or not-p at t (or:
For any possible state of affairs ω, either it is the case in ω that p or it is the case in ω that not-p).[4]

(The idea of introducing the notion of statability directly into the symbolism is due to J.L. Mackie). In the matrix for Q, Sp = 1 so long as there are no 2's in p, while if p has any 2's, Sp has 2's where p has them and 3's everywhere else. The L of Q is then definable in terms of M and S, since Lp = 'p is always statable and never false' (or 'p must be statable and cannot be false'). And the following M-S logic would appear to have Q for its M-L portion:- For S, lay down the following two rules (reflecting the fact that a proposition is statable if and only if all its constituent propositions are):-

RS1 : ⊢CSαSp, where p is any variable in α.

RS2 : ⊢CSpCSq....Sα, where p, q, etc. are all the variables in α.

[4] We have left Prior's punctuation unchanged.

For M, we need modifications of the following postulates (for subjoining to the propositional calculus) for S5:

M1 : $\vdash C\alpha\beta \rightarrow \vdash CM\alpha\beta$, provided β is fully modalised (i.e. each of its variables falls within the scope of some M)

M2 : $\vdash C\alpha\beta \rightarrow \vdash C\alpha M\beta$. (or: the axiom CpMp).

Df.L : Lp = NMNp.

M2 does not in fact require modification; the modifications required in the others would appear to be

SM1 : $\vdash C\alpha\beta \rightarrow \vdash CSpCSq....CM\alpha\beta$, where β is fully modalised (i.e. each of its variables falls within the scope of some M or S[5]), and p, q, etc. are all the variables (if there are any) which are in β but not in α.

Df.L : Lp = KSpNMNp.

The functor S may also be used to simplify and clarify certain extensions of Q due to E.J. Lemmon, namely

(i) Q + $\vdash$LCpp (= S5). Matrix : Q's, with all values containing 2's removed.

(ii) Q + $\vdash$NLp. Matrix : Q's, with all values <u>not</u> containing 2's removed.

(iii) Q + $\vdash$CMpp. Matrix : Q's, with every value which contains both 1's and 3's removed.

(iv) Q + $\vdash$CLMpp. Matrix : Q's, with every value which contains both 1's and 3's but not 2's removed.

In (i), both L and M are still genuinely modal functors, but we do not need both as primitives (can define L as NMN or M as NLN). In (ii) L is destroyed as

[5] Here 'or S' has been added in handwriting.

a modal functor (Lp becomes contradictory) but M is not. In (iii) M is destroyed as a modal functor (Mp becomes equivalent to the simple p) but L is not. The meaning of (iv) will be made clear later; it is contained both in (ii) and in (iii) but not in (i), and it cannot be combined with (i) without destroying both *L* and *M* as modal functors. (Q + ⊢LCpp + ⊢CLMpp → both ⊢CMpp and ⊢CpLp).

To obtain M-S systems with (i) – (iv) as their M-L fragments, make the following extensions to the postulates given above:-

For (i), add ⊢Sp (or define Sp as Cpp)

For (ii), add ⊢NSp (or define Sp as NCpp)

For (iii), add ⊢CMpp (or define Mp as p)

For (iv), add ⊢CSpCMpp.

The addition of ⊢Sp collapses SM1 (by a series of detachments) to M1, and KSpNMNp (= Lp) to the plain NMNp, giving the postulates of S5, and exhibiting S5 clearly as the result of assuming that all beings are necessary beings. (ii) results from the contrary assumption that no beings are necessary beings (and was designed by Lemmon to express this assumption). In (iii), the addition makes Lp (KSpNMNp) equivalent to KSpp ('p is always statable, and at present true'), the laws of this queer 'necessity' being deducible from this equivalence and RS1 and 2. (SM1 being now redundant). And now we see what (iv) is – it is a modal system which, unlike (i) and (ii), allows for both necessary and contingent beings; but it assumes (for it is a simple matter to deduce ⊢CSpCpLp in (iv)) that no proposition which is about necessary beings only, can be contingent; all such propositions are either necessary or impossible (though propositions which are about both necessary and contingent beings – e.g., perhaps, '9 is the number of planets in the solar system' – may be contingent).

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Note on the Interpretation of SM1. – Where there are no variables at all in β that are not in α, there are no Sp antecedents and the conclusion of the rule is simply as in M1.

Note 2. – The above M-S calculus may be developed within the M-L calculus Q as well as vice versa, for Sp is definable in terms of the L of Q as LCpp.

5. Prior's addendum

In Prior's Nachlass in the Bodleian Library, there is a version of the 1957 notes marked at the top 'New stuff' in Prior's handwriting.[6] The 'new stuff', which is not dated, consists of material he has added directly beneath Note 2. Its chief point of interest is a proposed axiomatization of Q using M and L.

This axiom system initially appeared in a draft letter Prior scribbled out in 1957, on the last day of November, while on board the M.V. Wanganella, on his way to Australia to attend the East-West Philosophers' Conference in Canberra (Prior 1957b). The draft begins 'Dear John'.[7] It mentions an axiomatic system of Lemmon's, devised in Oxford in 1956 at the time of Prior's lectures on Q. (*Time and Modality* was based on Prior's 1956 John Locke lectures at Oxford.) Lemmon conjectured that his system axiomatises Q. The system takes *L* and *M* as primitive (there is no *S*). In a later letter, Lemmon described how he came up with the system: 'it wasn't more than thinking of some matrices and guessing at axioms' (Lemmon 1958). In the Wanganella draft, Prior improved on Lemmon's system, setting out the proposed axiomatisation of Q that then appears in the 'new stuff'. He said to Lemmon in the Wanganella draft: 'This is two axioms less than your suggested set … and is at least as strong' (Prior 1957b: 4).

6 Bodleian Library, Oxford, MS. 12189/1.
7 Mary Prior incorrectly wrote 'John Mackie' at the top of the draft before depositing it in the Bodleian Library.

The 'new stuff' is as follows. 'Df.S' refers to the definition given at the end of Note 2, above.

With this Df.S, the following postulates in M and L are equivalent to the above ones, suggested for Q, in M and S:-

QM1 : ⊢Cαβ → ⊢CMαβ, if β is fully modalised, (i.e. each of its variables is within the scope of some M or L) and has no variables not in α.

M2 : As for S5.

RL : ⊢Cαβ → ⊢CLαLβ, if all variables in β are in α.

Axioms : 1. CKLpLqLKpq; 2. CLpNMNp; 3. CLCppCNMNpLp.

<u>Note 3</u>. – Calculus (iii) above has the following finite characteristic matrix:-

	K	1	2	3	4	N	L	S	M
*	1	1	2	3	4	4	1	1	1
*	2	2	2	3	3	3	3	3	2
	3	3	3	3	3	2	3	3	3
	4	4	3	3	4	1	4	1	4

which is exactly fitted by the following postulate-sets (subjoined to PC):-

(a) in L (Sp = LCpp)
(Mp = p)

RL as above; and
1. CKLpLqLKpq
2'. CLpp
3'. CLCppCpLp

(b) in S (Lp = KSpp)
(Mp = p)

RS1, RS2 as above.

To demonstrate this, translate 'p = 1' as Lp (or KSpp), 'p = 2' as KNLpp (or KNSpp), 'p = 3' as KNLNpNp (or KSpNp), and 'p = 4' as LNp (or KSpNp), and prove implications corresponding to all the equations in the matrix (e.g. 'L2 = 3' means 'If p = 2, Lp = 3', i.e. 'If KNLpp then KNLN(Lp)N(Lp)', i.e. CKNLppKNLNLpNLp).

RL, 1, 2′, 3′ are also in the weaker calculus (iv), which therefore has the same L-fragment as (iii). Add CLqLCpp to RL, 1, 2′, 3′, or replace 3′ by 3′′. CLqCpLp, and RL and 1 become superfluous. The resulting calculus, with Mp for NLNp instead of for p, is equivalent to the L-modal system of Lukasiewicz without the variable functor δ.

6. Łukasiewicz on Q

Before Prior delivered his John Locke lectures, he sent a typewritten version of some of the material to Łukasiewicz in Dublin. Łukasiewicz responded by letter (on 20 January 1956) with some comments about Q and other matters, despite being mortally ill in hospital. He said:

> [I]t is evident that your 6-valued matrix of Q (and, as I suppose, your infinitely-valued matrix) does not verify the rule of detachment. This is a defect that cannot be amended, and I think therefore that your system Q is defective too.
>
> (Łukasiewicz 1956: 4)

Prior discusses detachment (modus ponens) in *Time and Modality* and quotes from Łukasiewicz's letter. He points out that the 'awkwardness about the verification of the rule of detachment' arises because both *Cpq* [$p \rightarrow q$] and *p* may have a designated value when *q* has an undesignated value (Prior 1957a: 45). This situation occurs if *p* is unstatable and *q* is false; in this case *Cpq* is also unstatable, since an implication is unstatable if its antecedent is unstatable, regardless of the value of the consequent.

Prior spells out his response to Łukasiewicz, stating the following form of detachment (where α and β are metavariables ranging over formulae of Q):

> if the formula α is so constructed that every proposition of this form has a designated value, and the formula $C\alpha\beta$ (where α is as before) also has this property, then β has this property.
>
> (Prior 1957a: 46)

We will call this $\mathrm{Det}_{\vDash}$, to distinguish it from what we will call $\mathrm{Det}_{\vdash}$, the corresponding rule of derivation, which Prior uses in axiomatic derivations in Q (e.g., Prior 1964: 216). $\mathrm{Det}_{\vdash}$ is: if $\vdash C\alpha\beta$ and $\vdash\alpha$, then $\vdash\beta$.

Prior then explains that $\mathrm{Det}_{\vDash}$ 'can be shown to hold in Q' and gives a proof (Prior 1957a: 46). This proof might not have satisfied Łukasiewicz, however. In his letter, he introduced a matrix he called M and observed

> The system defined by the matrix M is functionally not complete, i.e. not all functors which may be constructed by C and N … are definable by C and N. If we want to have a complete system, we must add new functors.
>
> (Łukasiewicz 1956: 5)

By 'functionally complete', Łukasiewicz meant that 'all functions possible in the system' are definable in it (Łukasiewicz 1938: 97). Functional completeness was a preoccupation of the Warsaw logicians. In their many-valued logics, each additional matrix-value brought an exponential increase in the number of possible functions. Łukasiewicz held that 'if we wish to have a logic that is useful … in all cases, we have to strive to construct … systems in which all functions possible in the system are definable' (ibid.: 95). Anything less than a complete system was by his lights less than perfect: 'Partial systems are incomplete, therefore imperfect', he said (ibid.: 95). Q is not functionally complete.

Łukasiewicz's own three-valued system in *C* and *N* was initially incomplete (ibid.: 96-97)—until Słupecki added a new unary functor *T*, with *Tp* taking the third value (½) for all *p* (Słupecki 1936). Słupecki then axiomatized the extended system, by adding two axioms for *T* to Wajsberg's earlier axiomatization of Łukasiewicz's incomplete system (Słupecki 1939, Wajsberg 1931). Prior knew about Słupecki's *T*. In a review article published in July 1957, just a few months before the date of the 1957 notes, he wrote of

> Słupecki's 1936 completion of Wajsberg's 3-valued C - N postulates with *CTpNTp* and *CNTpTp*, where *Tp* = ½ whatever *p* is.
>
> (Prior 1957c: 410)

His own addition of *S* to Q was analogous to Słupecki's addition of *T* to the Łukasiewicz system. Might Łukasiewicz's comments on Q in his letter, together with Słupecki's use of *T*, have served as the inspiration for the approach that Prior followed in the 1957 notes? This is an interesting conjecture, but the evidence is purely circumstantial.

Łukasiewicz ended his comments on Q with a caution, quoted by Prior: 'it is always dangerous to use such matrices as models of systems' (Łukasiewicz 1956: 6, Prior 1957a: 54). Immediately after this quotation comes the closing paragraph of Prior's chapter on 'The System Q'. He says:

> It is fairly clear that our matrix for Q, like the simpler matrix M in Łukasiewicz's letter, only verifies the rule of detachment by being functionally incomplete.
>
> (Prior 1957a: 54)

And on that rather inconclusive note, the discussion of Q in *Time and Modality* came to an end. In our next section Robert Bull—interviewed in 2023—revisits Łukasiewicz's remarks on Q.

7. Bull on Q

COPELAND: When did you encounter Prior's 1957 notes?

BULL: That was in 1958. I first encountered Prior himself in 1957 when—on my father's advice—I took his Philosophy 1 classes for the B.Sc. Arthur was just back from his study leave in England, fizzing with excitement over corresponding with the friends he'd made there—John Lemmon and Carew Meredith especially. Then, in 1958, I was in his senior logic group. This was a research group for staff and students—Mary Prior attended too. The next year, 1959, I was doing my M.Sc and Arthur had left for Manchester, but had arranged for me to give some lectures in his Advanced Logic course. Then in 1960 I too was at Manchester, doing another M.Sc, this time in group theory. It was in the 1958 senior logic group that Arthur told us about Q and gave out the notes—that was the basis for my own work on Q, culminating in my completeness result.

COPELAND: *Time and Modality* also played some role in your investigation of Q?

BULL: Yes. I was definitely well aware of *Time and Modality* by the time I wrote the '64 article reporting my completeness result, and I referred to *Time and Modality* several times in the article.

COPELAND: Some of Łukasiewicz's comments on Q puzzle me. Is the functional incompleteness of Prior's matrix for Q as significant as Łukasiewicz seemed to think?

BULL: Functional completeness was very much a Polish thing, which I for one never took seriously. In fact I didn't give a brass razoo for functional completeness—beyond some nice five-finger technical exercises. Dear old Meredith was really good at these. I would meet him at sessions of the British Logic Association—where he would tell me that I should have been a priest, in between shouting me endless beer. He apparently considered the purchasing of beer to be one's duty with impoverished research students.

COPELAND: What do you think of Łukasiewicz's warning about using a functionally incomplete matrix as a model of Q?

BULL: Doesn't make any sense to me.

COPELAND: He seems to have been worried about contradiction creeping in. He was talking in particular about what he calls 'irregular' matrices, which are ones that do not verify detachment, and he says: 'It seems to me that any irregular matrix contains potentially contradiction and I think for this reason that it is always dangerous to use such matrices as models of systems' (Łukasiewicz 1956: 6).

BULL: Q can't be inconsistent because we have a model.

COPELAND: Might Łukasiewicz's concerns about functional incompleteness have played some role in the motivation for adding *S* to Q?

BULL: Adding in *S* would certainly give one far more of the functions on the 3-valued table. It also enables one to express the difference between 1 and 2 as designated values. Q with *S* really is a much better system.

COPELAND: What about Łukasiewicz's claim that Q is defective because detachment fails?

BULL: He is simply wrong, for languages without added propositional constants—and in fact also wrong for languages with propositional constants complying with Mary's restriction, which Arthur stated on p. 47 of *Time and Modality*. Look, we have to be careful. In Q, as the system is with no added propositional constants, we have the derivable formula $\vdash CpCCpqq$ [$p\rightarrow((p\rightarrow q)\rightarrow q)$]. And we also have a rule: 'If $\vdash\alpha$ and $\vdash C\alpha\beta$ then $\vdash\beta$', where α and β are metatheoretic symbols standing for formulas. We can substitute formulas α, β for the variables *p*, *q*, etc. And we *don't* have a rule: 'If $\vdash p$ and $\vdash Cpq$ then $\vdash q$'. For a start, if $\vdash p$ then $\vdash\alpha$ for every formula α, by substitution. If $\vdash p$, then substituting *Np* for *p* gives $\vdash Np$, so making the system inconsistent.

COPELAND: Arthur mentions the case where both *Cpq* and *p* have a designated value, but *q* has an undesignated value, and he says this leads to 'a certain awkwardness about the verification of the rule of detachment' (Prior 1957a: 45). What's awkward about it?

BULL: So what are the effects if *p* has value 2 and *q* has value 3? *Cpq* has value 2 rather than 3. $\models CpCCpqq$ [$p\rightarrow((p\rightarrow q)\rightarrow q)$] is still OK, because the *p*'s make the value of the whole thing 2. And in *Time and Modality* Arthur claimed that the *rule* is also OK (Prior 1957a: 46).

COPELAND: That's the million dollar claim. How would you convince, or try to convince, Łukasiewicz that the rule of detachment ($\mathrm{Det}_{\models}$) holds in Q?

BULL: Arthur and Alan Ross Anderson each claimed to show that detachment holds in the system. Arthur's version of the demonstration is on p. 46 of *Time and Modality*, in a footnote, and he mentions Anderson's in the same footnote. This *should* have satisfied Łukasiewicz, despite his concerns about functional completeness—but he died before Prior produced this argument, so we'll never know what he would have said about it.

COPELAND: How does the argument go?

BULL: You can put it this way, which isn't quite the way Arthur put it in his footnote—he made a bit less use of the formalism. Consider formulas α and β such that $\models \alpha$ and $\models C\alpha\beta$ and not $\models \beta$. Let $q_1, \ldots, q_n$ be the variables in β, and let $p_1, \ldots, p_m$ be the variables, if any, which are in α but not in β. Since not $\models \beta$, there is a valuation of $q_1, \ldots, q_n$ which gives β the value 3. Use the same valuation, adding any values other than 2 to $p_1, \ldots, p_m$. Since $\models C\alpha\beta$, this must give α the value 3, so contradicting $\models$

α. Łukasiewicz was forgetting that talking about $\vDash$ gives one freedom to pick and choose one's evaluation!

COPELAND: Turning to Arthur's axiomatisations, there is no statement of a rule of detachment ($\mathrm{Det}_{\vdash}$) in the systems he presents in the 1957 notes, but he clearly considers it to be present, since he speaks of 'a series of detachments'. It's the same in his 1959 paper. All we get from him, there and in the notes, is the terse remark that his new axiomatic apparatus is for 'subjoining' to the propositional calculus. I assume he thought that because detachment is in the propositional calculus, it is available in Q?

BULL: Yes, detachment is a rule of PC, so it's included in the extended system. But, of course, you need to check that the rule is still valid in the extended context. That's essentially what Arthur's argument in his footnote does. In my '64 paper I simply referred readers to Arthur's footnote for 'the verification of the rule of detachment' (Bull 1964: 214).

COPELAND: So in the end all this is part of proving for Q that if $\vdash\alpha$ then $\vDash \alpha$?

BULL: Yes, one has two systems running in parallel, the $\vdash$ system, the derivable formulas, and the $\vDash$ system, the valid formulas. One shows by induction on the length of derivations that if $\vdash\alpha$ then $\vDash \alpha$.

COPELAND: Is there any indication that Arthur constructed a proof of that nature?

BULL: He gives some further rules for the modal operators of Q, on page 47 of *Time and Modality*, and comments that these are 'obviously verified by our infinite matrix'—which is fair enough, as there are no tricks involved here. So yes, putting the pieces together he has got the essentials of a demonstration that if $\vdash\alpha$ then $\vDash \alpha$.

COPELAND: Would he have done the induction on the length of derivations?

BULL: He wouldn't have spelled it out. That wasn't Arthur's style!

COPELAND: What about the converse of if $\vdash\alpha$ then $\vDash \alpha$?

BULL: If $\vDash \alpha$ then $\vdash\alpha$—that's more difficult! And of course the two together are required for completeness.

COPELAND: Arthur didn't attempt a completeness proof?

BULL: No sir! And I doubt he could have proved completeness with the rules of *Time and Modality* and the 1957 notes.

COPELAND: But you proved completeness, didn't you?

BULL: Yes. And a bloody awful proof it looks. For a start, my '64 paper used more complicated rules for the modal operators—which Arthur then followed in his 1964 paper. These needed to be more general than Arthur's original ones, in order to push the proof through. And there are problems arising from Q's having the two distinct but related necessity operators, so that the strong **L** can influence the weak L, as I wrote them. Using my new rules, I then shoved any given formula into formulas in a complicated normal form, which is where it all starts to get ugly. Finally, I cobbled up a valuation that rejects any normal form which can't be derived. Whew.

COPELAND: In Krister Segerberg's preamble to his completeness proof—which he devised a few years after yours—he complained that, despite what he called the 'technical ingenuity' of your treatment, your completeness proof 'does not yield any semantical insight into Q' (Segerberg 1967: 68). Would you disagree?

BULL: He's quite right. His Scott-style semantic proof was more natural. I was working in the older tradition of matrix semantics, as was Arthur of course. This algebraic approach does have its place in logic—especially in pioneering explorations like Arthur's investigation of Q.

COPELAND: Last question. Do you know why Arthur chose Q as the name of his system?

BULL: No. And my impression is that I never did. With Arthur, never underestimate whimsy.

COPELAND: It's interesting that a few years previously, in 1953, he used 'Qp' to mean 'It is contingent that p' (Prior 1953: 322). Perhaps this was because the more obvious choices 'C' and 'K' were already in play. Could it be the system Q for 'Contingent'? A modal logic for contingent beings?

BULL: That does have the virtue of making Priorish sense. I guess it's one more thing we'll never be able to clinch.

8. Conclusion: Developments following the 1957 notes

The 1957 notes were an important step on the path to provably complete axiomatisations of Q.

Prior's 1959 paper 'Notes on a Group of New Modal Systems' was largely based on the 1957 notes. It begins by posing the open problem of finding an axiomatisation of Q. After a short description of Q's infinite matrix, Prior states 'I do not know of any set of postulates for which this matrix has been proved to be characteristic' (Prior 1959: 122). He then presents Lemmon's previously mentioned axiom system (Section 5) and reports Lemmon's conjecture that this system is Q. Next, Prior describes (what we are calling) Q_{MS}, saying that this system 'express[es] very directly indeed the basic intuitions underlying the Q matrix' (ibid.: 123). He claims a proof that Lemmon's system and Q_{MS} are equivalent, and offers this equivalence as evidence for Lemmon's conjecture. The equivalence, Prior says, 'seems to give considerable added weight to this conjecture' (ibid.: 123). However, he displays only some of the needed derivations, contenting himself with stating that the remainder are 'fairly obvious' (ibid.: 124).

Arguments of this type are of course no substitute for a completeness proof. In any case, Prior's claimed 'proof' of equivalence was erroneous. The problem lay with one of the cases he brushed aside as 'fairly obvious'—later realising that he had 'no way of proving' it (Prior 1964: 217). The axiomatisation problem continued to remain open until 1963, when Bull made his crucial contributions (Bull 1964).

Prior seized on Bull's result, and his paper 'Axiomatisations of the Modal Calculus Q', published alongside Bull's, contains his demonstration that all theses of Bull's system are theses of Q_{MS}. Prior says this result establishes 'the sufficiency of [Q_{MS}] for Q' (Prior 1964: 215). He did not explain exactly what he meant by 'sufficiency', but no doubt his focus was the claim from the 1957 notes that Q_{MS} has 'Q for its M-L portion'. Prior's demonstration shows that Q_{MS} has *at least* Q for its M-L portion, and Bull has pointed out that the argument can in fact be strengthened to give the claim from the notes:

> Yes, he's shown that everything in Q is in Q_{MS}. But Q_{MS} without *S* is still verified by the model for Q, so it must be exactly the same as Q. For since Q is complete for the model, anything in Q_{MS} (without S) that wasn't in Q would be rejected by the model.

By the time of *Past, Present and Future*, Prior was announcing that Q_{MS} is complete. He reported that, as a 'corollary' of Bull's completeness result, 'it was possible to prove completeness for some simpler postulates which I had put forward tentatively', and he is referring here to the presentation of Q_{MS} in his 1959 paper. We have not been able to locate this completeness proof. Prior gave no information concerning when it was devised or by whom. It was, though, not Bull's work. In any case, extending Bull's proof would have been tricky, because of the impact *S* would have on his procedure for constructing normal forms: 'Chucking *S* in would have pulled that around all over the place—better to start again with some different method of proof, I would have thought', Bull says. We leave this detail of the missing completeness proof as a small mystery.

Bull's pioneering completeness result was followed by Segerberg's (Segerberg 1967). His elegant Scott-style semantical proof that Q is the M-L fragment of his own axiomatic system HS5 cast new light on Q.[8]

In 1968, Prior summed up the state of play concerning axiomatisations of Q:

> Some time ago I proposed, and gave a matrix for, a modal system Q … This system has since been axiomatized by R. A. Bull, K. Segerberg and myself.
>
> (Prior 1968b: 183)

It is noticeable that Lemmon's conjectured axiomatisation, and Prior's improved version of it in the Wanganella draft and in the 'new stuff', were no longer listed as contenders. Of the three listed axiomatisations, Q_{MS} was certainly the most important as far as Prior was concerned. His frequent presentations of Q_{MS}, in contrast with his quite sparing mentions of the Bull and Segerberg systems, leave no room for doubt that his preferred axiomatisation of 'the true modal logic'—as he described Q towards the end of his life (Prior & Fine 1977: 104)—was the system that first saw the light of day in his fertile 1957 notes.

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8 For some critical comments by Fine, see Prior & Fine 1977: 148-149.

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