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Analytic Proof-Theory for Prior's System Q: First Steps

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Abstract

Arthur Prior introduced the three-valued modal logic called the Q system. A few axiom systems for the Q system can be found in the literature, but no analytic proof-theory in the form of tableau-, sequent- or natural deduction systems. In the present paper we demonstrate how to turn a formal semantics for the Q system into a tableau system, whereby we provide a proof system that is suitable for actual reasoning.

Keywords: Arthur Prior, System Q, tableau systems.

1 Introduction

Arthur Prior's aim with system Q was to give a modal logic for contingent beings. The guiding idea in the Q system is that a proposition that refers to non-existent individuals does not have a classical truth value. Thus, in this system, a proposition can either be true, false, or what Prior called unstatable, meaning that the proposition in question is not statable in the actual world or at the present time, that is, if it is about non-existing individuals. Prior described system Q as 'the true modal logic'.

Prior considered versions of the Q system in a number of publications, in particular in chapter V of the book Prior (1957). See also the book Fine and Prior (1977). Since then, a number of papers have been published about the system, recent examples are Copeland and Markoska-Cubrinovska (2023); Badie (2023). A few papers have considered Hilbert-style axiom systems for the Q system, including Segerberg (1967); Akama and Nagata (2007). However, compared to the voluminous literature on many-valued modal logics in general, the literature on the Q system is modest. In line with that observation, and to the best of our knowledge, no analytic proof-theory in the form of tableau-, sequent- or natural deduction systems have been given for the Q system. Such systems are meant for actual reasoning, whereas axiom systems are usually of a more foundational interest. The present paper will take the first steps in the development of a tableau system for the Q system.

A leading figure in the development of tableau systems was Jaako Hintikka, see Hintikka (1955). A milestone in the later development of tableau systems is Melvin Fitting's book Fitting (1983). See the handbook D'Agostino et al. (1999) for many more details. Hintikka made the following remarks on the idea behind tableau systems.

> ... the typical situation is one in which we are confronted by a complex formula (or sentence) the truth or falsity of which we are trying to establish by inquiring into its components. Here the rules of truth operate from the complex to the simple: they serve to tell us

what, under the supposition that a given complex formula or sentence is true, can be said about the truth-values of its components. (Hintikka (1955), page 20)

Thus, the idea is to mimic the recursive truth-conditions in a model-theoretic semantics, whereby a formula is broken down into its components. A very brief introduction to tableau systems can be found in the appendix of the present paper.

Our starting point is the above mentioned paper Akama and Nagata (2007) that gives a Hilbert-style axiom system for system Q, which is proved to be sound and complete wrt. a Kripke-style model-theoretic semantics. The Q system given in Akama and Nagata (2007) is based on the modal logic S5 where there is no accessibility relation (or equivalently, the accessibility relation is the universal relation). In the present paper we shall demonstrate how to turn (a version of) the formal semantics of Akama and Nagata (2007) into a tableau system. This is in line with the fundamental idea of Fitting's prefixed tableau systems Fitting (1983) and Dov Gabbay's labelled deductive systems Gabbay (1996) which is to prefix formulas in derivations by metalinguistic indexes, or labels, with the aim of regulating the proof process.

2 What is the Q system?

In this section we give the formal syntax and semantics of the Q system, taken from the paper Akama and Nagata (2007), which in turn follows the book Fine and Prior (1977). The syntax for formulas is obtained by extending the the syntax for ordinary propositional modal logic with a unary operator S, called the *statability operator*. Formulas are built in the usual way using negation $\neg$, conjunction $\wedge$, the modal possibility operator $\Diamond$, and the statability operator S. It is assumed that a countably infinite set $PROP$ of propositional symbols is given, ranged over by the metavariables p, q, r, $\ldots$. The set of formulas is denoted FOR. The metavariables $\phi, \psi, \theta, \ldots$ range over formulas.

The connectives disjunction and implication are defined by the standard conventions that $\phi \vee \psi$ is an abbreviation for $\neg(\neg\phi \wedge \neg\psi)$ and $\phi \rightarrow \psi$ is an abbreviation for $\neg\phi \vee \psi$. In system Q there is a necessity operator $\Box$ defined by $\Box\phi$ being an abbreviation for $S\phi \wedge \neg\Diamond\neg\phi$. Thus, contrary to ordinary modal logic, the operators $\Box$ and $\Diamond$ are not dual. Intuitively, $S\phi$ says that "ϕ is statable in all worlds" and $\Diamond\phi$ says that "ϕ is statable and it is true in some world" so $\Box\phi$ says that "ϕ is true in all worlds". Of course, in the temporal version of the S operator, 'worlds' has to be replaced by 'times' as appropriate.

As suggested earlier, Prior's Q system involves three truth-values, namely true (denoted 1), false (denoted 0), and unstatable (denoted 2), the latter meant for interpretation of propositions not statable in the actual world or at the present time.[1] The three-valued logic underlying the formal semantics for Q given in the paper Akama and Nagata (2007) (and also the temporal version given in Akama et al. (2008)) is Kleene's weak three-valued logic, which the authors of the two papers argue is in line with Prior's ideas. We now define models.

[1]The paper Akama and Nagata (2007) uses -1 for unstatable, but to save notation, we follow some other authors in using 2 instead.

Definition 1 *A* model *for the Q system is a tuple* $(W, \texttt{stat}, V)$ *where*

1. *W is a non-empty set;*
2. *V is a function that to each pair in $PROP \times W$ assigns an element of $\{0, 1, 2\}$; and*
3. $\texttt{stat}$ *is a relation on $FOR \times W$ such that $\texttt{stat}(\phi, w)$ if and only if for any propositional symbol p occuring in ϕ, it is the case that $V(p, w) \neq 2$.*

The elements of W are called worlds *and the relation* $\texttt{stat}$ *is called the* statability relation*.*

Given a model $(W, \texttt{stat}, V)$, the function V is by induction extended[2] to a function that to each pair in $FOR \times W$ assigns an element of $\{0, 1, 2\}$.[34]

$$
\begin{array}{rcl}
V(\neg\phi, w) = 1 & \text{iff} & \texttt{stat}(\phi, w) \text{ and } V(\phi, w) \neq 1 \\
V(\neg\phi, w) = 0 & \text{iff} & \texttt{stat}(\phi, w) \text{ and } V(\phi, w) \neq 0 \\
V(\neg\phi, w) = 2 & \text{iff} & \text{otherwise} \\
V(\phi \wedge \psi, w) = 1 & \text{iff} & V(\phi, w) = 1 \text{ and } V(\psi, w) = 1 \\
V(\phi \wedge \psi, w) = 0 & \text{iff} & \texttt{stat}(\phi, w) \text{ and } \texttt{stat}(\psi, w) \\
 & & \text{and } (V(\phi, w) = 0 \text{ or } V(\psi, w) = 0) \\
V(\phi \wedge \psi, w) = 2 & \text{iff} & \text{otherwise} \\
V(\Diamond\phi, w) = 1 & \text{iff} & \texttt{stat}(\phi, w) \text{ and } \exists v \in W(V(\phi, v) = 1) \\
V(\Diamond\phi, w) = 0 & \text{iff} & \texttt{stat}(\phi, w) \text{ and } \forall v \in W(V(\phi, v) = 0) \\
V(\Diamond\phi, w) = 2 & \text{iff} & \text{otherwise} \\
V(S\phi, w) = 1 & \text{iff} & \forall v \in W(\texttt{stat}(\phi, v)) \\
V(S\phi, w) = 0 & \text{iff} & \exists v \in W(not(\texttt{stat}(\phi, v))) \\
V(S\phi, w) = 2 & \text{iff} & falsum
\end{array}
$$

A formula ϕ is *true* in a model $(W, \texttt{stat}, V)$ if and only if $V(\phi, w) = 1$ for any world $w \in W$ at which ϕ is statable, that is, such that $\texttt{stat}(\phi, w)$. A formula ϕ is *valid* if and only if ϕ is true in any model.

3 Tableaus for the Q system

At this point, we take it that the reader is already familiar with tableau systems. If not, the the reader is advised to consult the appendix where we sketch some basics of tableau systems for ordinary propositional logic. Alternatively, a brief introduction to tableau systems can be found in Chapter 1 of Priest (2008).

[2]The induction is slightly different from, but equivalent to, the induction in Akama and Nagata (2007).

[3]Our clause for $V(\phi \wedge \psi, w) = 0$ is obtained by correcting the corresponding clause in Akama and Nagata (2007), page 107, first column such that it is consistent with the text in the proof of Theorem 1, Akama and Nagata (2007), page 107, second column, lines 10–13 (showing that the conjunction of Akama and Nagata (2007) is the conjunction of Kleene's weak three-valued logic, in particular, that $\phi \wedge \psi$ is unstatable if either ϕ or ψ is unstable, that is, $\phi \wedge \psi$ is assigned the truth-value 2 if either ϕ or ψ has the truth-value 2). We note that the later paper Akama et al. (2008), page 221, makes use of the correct clause for conjunction.

[4]Note that the clause for $V(S\phi, w) = 2$ is $falsum$, which might seem odd, but in light of the fact that V is a mathematical function, the clause simply says that the value of the function V on the pair $(S\phi, w)$ is never 2, that is, it is always 1 or 0. More informally, a formula on the form $S\phi$ is never unstatable, that is, it is always true or false, depending on the specified conditions.

Now, our starting point is the formal semantics for the Q system given in the paper Akama and Nagata (2007), or to be more precise, the slightly reformulated version of the semantics given in Section 2. We turn each clause (truth-condition) in this formal semantics into a tableau rule (or in one case, no rule at all).

The formulas in the tableau rules are formulas of the Q system—the object language—decorated with metalinguistic machinery mimicking the formal semantics. This metalinguistic machinery involves what are called *labels*. We assume that a countably infinite set of labels is given. We let the metavariables $l, m, n \ldots$ range over labels. The intended interpretation of the labels are possible worlds in the formal semantics. We shall later introduce a function $\mathcal{N}$ from labels to worlds that takes care of this interpretation.

If ϕ is a formula of the Q system, then $V(\phi, l) = 1$, $V(\phi, l) = 0$ and $V(\phi, l) = 2$ are metalinguistic formulas that can occur in the tableau rules. Note that such a metalinguistic formula, for example $V(\phi, l) = 1$, is just a formal piece of syntax, but of course with an intended interpretation, namely a mathematical statement saying that the formula ϕ has the truth-value 1 relative to a world denoted l in a given formal semantics.

3.1 Turning the formal semantics into a tableau system

As we will explain in what follows, our turning the formal semantics into tableau rules hinges on Lemma 3 in Akama and Nagata (2007), page 107, saying that $\mathtt{stat}(\phi, w)$ if and only if either $V(\phi, w) = 1$ or $V(\phi, w) = 0$ (or equivalently, if and only if $V(\phi, w) \neq 2$). This is crucial since it allows us to turn statements about the truth or falsity of $\mathtt{stat}(\phi, w)$ into statements about the three-valued truth-value of $V(\phi, w)$. Turning the formal semantics of the statability operator S into tableau rules also hinges on Lemma 4 in Akama and Nagata (2007), page 107, saying that $V(S\phi, w) = 1$ if and only if $\forall v \in W(V(\phi, v) \neq 2)$, which is a direct consequence of Lemma 3.

Now, let us turn to the tableau rules. Each clause in the formal semantics of Section 2 is converted to exactly one (or in one case zero) of the tableau rules in figures 1, 2, 3 and 4 by taking as the premise the metalinguistic formula corresponding to the left hand side of the clause, and as the conclusion of the tableau rule, we take metalinguistic formulas corresponding to the right hand side of the clause, in some of the cases after reformulation based on combinatorial analyses. Recall that a vertical bar | in the conclusion of a tableau rule should be read as a disjunction, that is, the metalinguistic formulas separated by vertical bars should be added to the tableau as different branches. We describe this conversion procedure by considering each connective in the formal semantics.

Our tableau rules for the Boolean connectives negation $\neg$ and conjunction $\wedge$ can be found in Figure 1 and Figure 2 respectively. Loosely speaking, our rules for the Boolean connectives can be seen as tableau versions of what are called *3-labelled calculi* for three-valued propositional logics described in Greati et al. (2024) and elsewhere. We shall not go into details with the conversion procedure for negation and conjunction, but instead jump directly to the $\Diamond$ and S operators.

Our tableau rules for the possibility operator $\Diamond$ can be found in Figure 3. We now describe how the clauses for $\Diamond$ have been turned into tableau rules.

We begin with the clause for $V(\Diamond\phi, w) = 1$, that is:

$$V(\Diamond\phi, w) = 1 \text{ iff } \texttt{stat}(\phi, w) \text{ and } \exists v \in W(V(\phi, v) = 1)$$

cf. Section 2. We follow the general conversion procedure: The clause is converted into a tableau rule by converting the left hand side of the clause into a premise, namely $V(\Diamond\phi, l) = 1$, and the right hand side—reformulated as appropriate—into a conclusion. Our reformulation makes use of the observation that if the right hand side of the clause is true, then by Lemma 3 in Akama and Nagata (2007), we have two cases:

1. $V(\phi, w) = 1$ and $\exists v \in W(V(\phi, v) = 1)$, which together is equivalent to $V(\phi, w) = 1$.
2. $V(\phi, w) = 0$ and $\exists v \in W(V(\phi, v) = 1)$.

These two cases are reflected in the conclusion of the first tableau rule in Figure 3 where each case corresponds to one branch (note vertical bar | separating the conclusion into two parts). Note that the existential quantifier in the second case above is dealt with by introducing a new label m, analogous to the standard tableau rule for eliminating the existential quantifier in first-order logic. The clause for $V(\Diamond\phi, w) = 0$ is the following:

$$V(\Diamond\phi, w) = 0 \text{ iff } \texttt{stat}(\phi, w) \text{ and } \forall v \in W(V(\phi, v) = 0)$$

again cf. Section 2. If the right hand side of the clause is true, then by Lemma 3 in Akama and Nagata (2007), we have two cases:

1. $V(\phi, w) = 1$ and $\forall v \in W(V(\phi, v) = 0)$, which is inconsistent.
2. $V(\phi, w) = 0$ and $\forall v \in W(V(\phi, v) = 0)$, which together is equivalent to $\forall v \in W(V(\phi, v) = 0)$.

Since the first case is inconsistent, we only need to consider the second case, which is reflected in the conclusion of the second tableau rule in Figure 3. Note that the universal quantifier in the second case above is dealt with analogous to the standard universal instantiation rule in first-order logic, involving the label n that already occurs at the branch. In a similar—but slightly more involved—way we turn the clause for $V(\Diamond\phi, w) = 2$ into a tableau rule, see the third rule in Figure 3, with the metalinguistic formula $V(\Diamond\phi, l) = 2$ as premise.[5]

Our tableau rules for the for the statability operator S are obtained in a similar way, see Figure 4, but note that there is no rule for the metalinguistic formula $V(S\phi, l) = 2$ since for any formula ϕ, either $V(S\phi, w) = 1$ or $V(S\phi, w) = 0$. However, this is dealt with by the closure conditions, which we shall turn to below.

Having the syntactic machinery in place (formulas, metalinguistic formulas, tableau rules), we need the following definition which plays a crucial rule in the soundness result of the following subsection. A branch in a tableau is called *closed* if at least one of the following two conditions are satisfied.

1. A metalinguistic formula $V(\phi, l) = i$ occurs on the branch, such that also $V(\phi, l) = j$ occurs on the branch, where $i \neq j$.

[5] The tableau rule for $V(\Diamond\phi, l) = 2$ will not win a beauty contest, cf. Figure 3, but at this stage something that works takes priority over possible aesthetic qualities.

Figure 1: Tableau rules for negation

$$\frac{V(\neg\phi, l) = 1}{V(\phi, l) = 0} \qquad \frac{V(\neg\phi, l) = 0}{V(\phi, n) = 1} \qquad \frac{V(\neg\phi, l) = 2}{V(\phi, n) = 2}$$

Figure 2: Tableau rules for conjunction

$$\frac{V(\phi \wedge \psi, l) = 1}{V(\phi, l) = 1, V(\psi, l) = 1} \qquad \frac{V(\phi \wedge \psi, l) = 2}{V(\phi, l) = 2 \mid V(\psi, l) = 2}$$

$$\frac{V(\phi \wedge \psi, l) = 0}{V(\phi, l) = 0, V(\psi, l) = 0 \mid V(\phi, l) = 0, V(\psi, l) = 1 \mid V(\phi, l) = 1, V(\psi, l) = 0}$$

2. A metalinguistic formula $V(S\phi, l) = 2$ occurs on the branch.

A branch is called *open* if it is not closed. A tableau is called *closed* if all branches are closed.

3.2 Soundness

Before dealing with soundness, we need some basic definitions linking the syntactic machinery to the formal semantics. We say that a set of metalinguistic formulas is *satisfiable* if there exist a valuation V and a function $\mathcal{N}$ that maps labels to worlds, such that for any metalinguistic formula $V(\phi, l) = i$ in the set, it is the case that $V(\phi, \mathcal{N}(l)) = i$. A tableau branch is *satisfiable* if the set of metalinguistic formulas on it is satisfiable, and a tableau is *satisfiable* it it has a satisfiable branch.

Lemma 1 *If a branch of a tableau is satisfiable, and a tableau rule is applied to the branch, then at least one of the generated branches is satisfiable.*

Proof: The lemma follows from an inspection of the each of the tableau rules, cf. figures 1, 2, 3 and 4. □

Theorem 1 *(Soundness) If there exists a closed tableau with root metalinguistic formula $V(\phi, l) = 0$, then the formula ϕ is valid.*

Proof: Assume conversely that there exists a closed tableau τ with root $V(\phi, l) = 0$, but the formula ϕ is not valid. The formula ϕ is not being valid implies that there exists a model $(W, \mathtt{stat}, V)$ in which ϕ is not true. Thus, either $V(\phi, w) \neq 1$ for some world $w \in W$ such that $\mathtt{stat}(\phi, w)$, or for no world $w \in W$ it is the case that $\mathtt{stat}(\phi, w)$.

In the first case where $V(\phi, w) \neq 1$ for some world $w \in W$ such that $\mathtt{stat}(\phi, w)$, it is the case that $V(\phi, w) = 0$, by Lemma 3 in Akama and Nagata

Figure 3: Tableau rules for the $\Diamond$ modality

$$\frac{V(\Diamond\phi, l) = 1}{V(\phi, l) = 1 \mid V(\phi, l) = 0, V(\phi, m) = 1}\ *$$

$$\frac{V(\Diamond\phi, l) = 0}{V(\phi, n) = 0}\ \star$$

$$\frac{V(\Diamond\phi, l) = 2}{V(\phi, l)=2 \mid V(\phi, l)=0, V(\phi, m)=2, V(\phi, n)=0 \mid V(\phi, l)=0, V(\phi, m)=2, V(\phi, n)=2}\ *\star$$

$*$ The label m is new.
$\star$ The label n is any label on the branch.

Figure 4: Tableau rules for the S modality

$$\frac{V(S\phi, l) = 1}{V(\phi, n) = 1 \mid V(\phi, n) = 0}\ \star \qquad \frac{V(S\phi, l) = 0}{V(\phi, m) = 2}\ *$$

$*$ The label m is new.
$\star$ The label n is any label on the branch.

(2007) saying that $\mathtt{stat}(\phi, w)$ if and only if either $V(\phi, w) = 1$ or $V(\phi, w) = 0$. Let $\mathcal{N}$ be a function from labels to worlds such that $\mathcal{N}(l) = w$. Then the branch constituted by just the root metalinguistic formula $V(\phi, l) = 0$ is satisfiable, and hence, by Lemma 1, the tableau τ is satisfiable, contradicting it being closed.

In the second case where for no world $w \in W$ it is the case that $\mathtt{stat}(\phi, w)$, we let $\mathcal{N}$ be an arbitrary function from labels to worlds. Then by Lemma 3 in Akama and Nagata (2007), $V(\phi, \mathcal{N}(l)) = 2$, so the branch constituted by just the root metalinguistic formula $V(\phi, l) = 2$ is satisfiable, and hence, by Lemma 1, the tableau τ is satisfiable, contradicting it being closed. $\Box$

Let us have a look at an example tableau that shows that the formula $\Diamond S\phi \rightarrow S\phi$ is valid. This formula is taken as an axiom in the axiom system given in Akama and Nagata (2007) and it is also taken as an axiom in Fine and Prior (1977), cf. page 85 of that book. Since $\rightarrow$ is a defined connective, we instead consider the equivalent formula $\neg(\Diamond S\phi \wedge \neg S\phi)$, where all connectives can be dealt with by our tableau rules. To show the validity of that formula, we have to show that there exists a closed tableau with root metalinguistic formula $V(\neg(\Diamond S\phi \wedge \neg S\phi), l) = 0$, cf. soundness, that is, Theorem 1 above. Below is the desired tableau.

$$V(\neg(\Diamond S\phi \land \neg S\phi), l) = 0 \qquad \text{1.}$$
$$V(\Diamond S\phi \land \neg S\phi, l) = 1 \qquad \text{2. from 1}$$
$$V(\Diamond S\phi, l) = 1 \qquad \text{3. from 2}$$
$$V(\neg S\phi, l) = 1 \qquad \text{4. from 2}$$
$$V(S\phi, l) = 0 \qquad \text{5. from 4}$$
$$V(\phi, m) = 2 \qquad \text{6. from 5 } (m \text{ is new})$$

$$V(S\phi, l) = 1 \quad \Big| \quad V(S\phi, l) = 0 \qquad \text{7. from 3}$$
$$V(S\phi, n) = 1 \qquad \text{8. from 3 } (n \text{ is new})$$

$$V(\phi, m) = 1 \quad \Big| \quad V(\phi, m) = 0 \qquad \text{9. from 8 } (m \text{ is on branch})$$

The enumeration of the lines and the notation in the right-hand-side column is not a formal part of the tableau, but has been added to describe how the tableau was constructed. Note that there are three branches and they are all closed: The left-hand-side branch contains $V(\phi, l) = 0$ as well as $V(\phi, l) = 1$, the middle branch contains $V(\phi, m) = 2$ as well as $V(\phi, m) = 1$, whereas the right-hand-side branch contains $V(\phi, m) = 2$ as well as $V(\phi, m) = 0$.

Note how the tableau is built step by step in a mechanical way by breaking down the root metalinguistic formula $V(\neg(\Diamond S\phi \land \neg S\phi), l) = 0$ into smaller and smaller metalinguistic formulas, as indicated earlier, cf. in particular the quotation by Hintikka in Section 1.

4 Conclusion and further work

In the present paper we have proved that our tableau system is sound wrt. the semantics given in Section 2 (that is, Theorem 1), and given that the system mimicks the formal semantics so closely, we conjecture that it is also complete (that is, that the converse of Theorem 1 also holds). We leace completeness to further work.

Together with soundness, a completeness result would give us a semi-decision procedure for validity: A formula ϕ would then be valid if and only if there exists a closed tableau with root metalinguistic formula $V(\phi, l) = 0$, and if such a tableau exists, it can be found by a semi-decision procedure based on the standard computer science technique called dovetailing. The dovetailing technique can be used since there are only finitely many different ways in which rules can be applied to a given tableau. We shall leave this to future work.

The similarities between system Q and the modal logic S5 has been noted several places, going back to Prior (1957), pages 47–48, and Fine and Prior (1977), pages 84–115, and more recently for example also discussed in Akama and Nagata (2007). The reader aware of the similarities between Q and S5, and also knowing about the huge literature on problems with giving cut-free[6] proof systems for S5, might question our completeness conjecture above since

[6] See the appendix for a brief discussion of the cut rule.

our tableau system for Q does not involve anything like the cut rule. An account of these problems with S5 anno the year 2000 can be found in Braüner (2000), in fact this issue has continued to generate papers up until the present time. However, notice that the problems with cut-free systems for S5 applies to systems without metalinguistic machinery, that is, systems in the style of the tableau system for ordinary propositional logic given in the appendix. On the other hand, labeled systems for S5 usually allows cut-elimination as for example witnessed by the cut-free prefixed tableau system for S5 given in the handbook chapter Fitting (2007) (see an illuminating discussion of this issue at page 108 in this handbook chapter).

More generally, we remark that not only is a tableau system for Q of interest for its own sake, but providing such a system also paves the way for further investigations, for example: Can the tableau system be adapted to other versions of Q, for example the temporal version considered in Akama et al. (2008)? Also, it will also allow us to compare (and contrast) the Q system to other many-valued modal logics via their proof systems, cf. Fitting (2007).

Another more general line of further work is the following: In the context of ordinary modal logic, Prior introduced what we nowadays call nominals—propositional symbols that are true at exactly one world—in the form of a second sort of propositional symbols. Nominals are an essential ingredient of contemporary hybrid logic, cf. for example Areces and ten Cate (2007); Braüner (2021). In the book Fine and Prior (1977), pages 112–115, Prior briefly discusses the Q-system extended with nominals:

> Suppose we equate each instant (or world) with some always statable proposition which is true at that instant (in that world), and there only, . . . (Fine and Prior (1977), page 112)

So in contemporary terminology, Prior discussed a hybrid-logical version of the Q-system. Hybrid-logical versions of Q along these lines—with nominals as a second sort of propositional symbols—would be interesting to investigate in their own right.

However, Prior observed that nominals can also be defined in S5 extended with propositional quantifiers, without the need for a new sort of propositional symbols, see the accounts in Blackburn et al. (2020, 2023). That is, Prior observed that propositionally quantified S5 allows the definition of an operator Q[7] such that Qp says that the ordinary propositional symbol p is true at exactly one world, that is, p works as a nominal. In the book Fine and Prior (1977), page 115, Prior very briefly discuss this operator-based definition of nominals in the context of the Q system extended with propositional quantifiers. This issue is also touched on in Kit Fine's postscript in Fine and Prior (1977), see pages 148–153. This issue, and propositionally quantified Q more generally, calls for further investigation, including proof-theoretic aspects parallel to the tableau systems for hybrid logic with propositional quantifiers given in Blackburn et al. (2020, 2023).

[7] As far as we know, there is no direct connection between Prior's naming of this operator and his naming of the Q system.

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Appendix: Some basics of tableau systems

We in this appendix sketch some basics of tableau systems and fix terminology. Introductions to tableau systems can be found many different places; we in this appendix give a very brief introduction based on Section 3.1 of the book Braüner (2011). In what follows, we stick to tableaus for ordinary propositional logic.

A *tableau* is a well-founded tree in which each node is labelled with a formula and the edges represent applications of tableau rules. By applying rules to a tableau, the tableau is expanded, that is, new edges and formulas are added to the leaves. A tableau is displayed such that it grows downwards. Technically, premises and conclusions of tableau rules are finite sets of formulas and a tableau rule has one premise together with one or more conclusions. A requirement for applying a rule to a branch in a tableau is that all the formulas in the premise are present at the branch, and the result of applying the rule is that for each conclusion of the rule, the end of the branch is extended with a path containing a node for each of the formulas in the conclusion in question. A branch in a tableau is called *open* if for no formula χ occurring on the branch, it is the case that $\neg\chi$ also occurs on the branch. A branch is called *closed* if it is not open. A tableau is called *closed* if all branches are closed.

For an example of a tableau system, see the standard tableau rules for ordinary propositional logic in Figure 5. The rule $(\wedge)$ in Figure 5 has one conclusion, namely $\{\phi, \psi\}$, which has two formulas, thus, the result of applying this rule to a branch is that the branch is extended with one path containing two nodes which are labelled with respectively ϕ and ψ. On the other hand, the rule $(\neg\wedge)$ has two conclusions (note: separated by a vertical bar | in the rule), namely $\{\neg\phi\}$ and $\{\neg\psi\}$, thus, the result of applying this rule to a branch is that the branch is extended with two paths each containing one node, where one node is labelled with ϕ and one node is labelled with ψ. Intuitively, the vertical bar | in the conclusion of a tableau rule should be read as a disjunction.

As described earlier, the idea behind tableau systems is that tableau rules are applied to break down formulas into smaller formulas. This idea is blatantly violated by what is called the *cut* rule:

$$\frac{}{\phi \ \mid \ \neg\phi}\,(cut)$$

Note that this rule does not have any formulas in the premise, so it can be applied at any stage in building a tableau, that is, at any stage, a branch can be

Figure 5: Tableau rules for propositional logic

$$\frac{\neg\neg\phi}{\phi}\,(\neg\neg)$$

$$\frac{\phi \wedge \psi}{\phi, \psi}\,(\wedge) \qquad \frac{\neg(\phi \wedge \psi)}{\neg\phi \mid \neg\psi}\,(\neg\wedge)$$

extended with two paths each containing one node, where one node is labelled with ϕ and one node is labelled with $\neg\phi$. In this sense, the cut rule builds in the principle bivalence in the tableau system.

If a tableau system includes the cut rule, then this rule can most often be proved to be redundant, that is, the cut rule is what is called *admissible* in the system obtained by leaving out the cut rule. This means that for any closed tableau including applications of the cut rule, there exists a closed tableau with the same root formula, but without applications of cuts. This is for example the case with the tableau system for propositional logic given in Figure 5. For more on cuts in tableau systems, see for example Fitting (2007).